Double Dose of Double-Negation Translations

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Double-Negation Translation: Five Ws

The theory:

- automatic theorem proving: classical logic
- other logics existing: need for translations
- in particular: proof-assistants
- related to the grounds:
  - ★ cut-elimination for sequent calculus
  - ★ extensions to Deduction Modulo

The practice:

- a shallow encoding of classical into intuitionistic logic
- Zenon modulo's backend for Dedukti
- existing translations: Kolmogorov's (1925), Gentzen-Gödel's (1933), Kuroda's (1951), Krivine's (1990), $\cdots$

Double-Negation Translation: Five Ws

Objective, **minimization**:

- turns more formulæ into themselves;
- shifts a classical proof into an intuitionistic proof of the *same* formula.

Today:

- first-order (classical) logic
- the principle of excluded-middle
- intuitionistic logic
- double-negation translations
- minimization
- **if** you're still alive:
  - extension to Deduction modulo
  - semantic Double-Negation translations
  - cut elimination

Theorem Proving

What do we prove ?

[Definition] Formula in Propositional Logic

- atomic formula: $P, Q, \cdots$
- special constants: $\bot, \top$
- assume A, B are formulæ: $A \wedge B, A \vee B, A \Rightarrow B, \neg A$

Example: $P \Rightarrow Q, P \wedge Q, Q \vee \neg Q, \bot \Rightarrow (\neg\bot), \cdots$

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[Definition] Formula in First-order Logic

- atomic formula: $P(t), Q(t, u), \cdots$
- connectives $\wedge, \vee, \Rightarrow, \neg, \bot, \top$
- quantifiers $\forall$ and $\exists$. Assume A is a formula and x a variable: $\forall xA$, $\exists xA$

- new category: terms (denoted a, b, c, t, u) and variables (x, y).
  Example: $f(x), \quad g(f(c), g(a, c)), \quad \cdots$
- Example: $(\forall xP(x)) \Rightarrow P(f(a)), \quad \exists y(D(y) \Rightarrow \forall xD(x))$

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- a theorem/specification is usually formulated as: *assume A, B and C. Then D follows*.

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- examples:
  - $A \vdash A$ is a (hopefully provable) sequent
  - $P(a) \vdash \forall x P(x)$ is a (hopefully unprovable) sequent
  - $A, B \vdash A \wedge B$, $A \vdash$, $A \vdash \bot$

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  - ★ $P(a) \vdash \forall x P(x)$ is a (hopefully unprovable) sequent
  - ★ $A, B \vdash A \wedge B$, $A \vdash$, $A \vdash \bot$
- classical logic needs multiconclusion sequent

[Definition] Classical Sequent

A classical sequent is a pair of sets of formulæ, denoted $\Gamma \vdash \Delta$

- ★ the sequent $A, B \vdash C, D$ must be understood as: *Assume A and B. Then C or D*

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How do we prove ?

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$$\frac{\text{premiss/antecedent} \qquad \text{premiss/antecedent}}{\text{conclusion/consequent}} \uparrow$$

read this way, please

- in order for the consequent to hold $\cdots$
- $\cdots$ we must show that the antecedent(s) hold

Endless process ?

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The axiom rule	The $\Rightarrow_R$ rule
$\dfrac{}{A \vdash A}\,\text{ax}$	$\dfrac{A \vdash B}{\vdash A \Rightarrow B}\Rightarrow_R$

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The real axiom rule	The real $\Rightarrow_R$ rule
$\dfrac{}{\Gamma, A \vdash A, \Delta}\ \text{ax}$	$\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$

- First example of proof: $\dfrac{\dfrac{}{A \vdash A}\ \text{ax}}{\vdash A \Rightarrow A}\ \Rightarrow_R$

The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta} \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee_R$$

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$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists_L \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \forall_R$$

Basic Examples

- commutativity of the conjunction:

$$A \wedge B \vdash B \wedge A$$

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- an alternative proof:

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- an alternative proof:

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- this is an example of the liberty allowed by Sequent Calculus
- excluded-middle:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{\vdash A, \neg A}\,\neg_R}{\vdash A \vee \neg A}\,\vee_R$$

More interesting examples

- uniform continuity implies continuity:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x,y) \vdash P(x,y)}\ \text{ax}}{P(x,y) \vdash \exists y P(x,y)}\ \exists_R \text{ (with } y)}{\forall x P(x,y) \vdash \exists y P(x,y)}\ \forall_L \text{ (with } x)}{\forall x P(x,y) \vdash \forall x \exists y P(x,y)}\ \forall_R \text{ (} x \text{ fresh)}}{\exists y \forall x P(x,y) \vdash \forall x \exists y P(x,y)}\ \exists_L \text{ (} y \text{ fresh)}$$

- the converse is fortunately not provable:

$$\dfrac{\dfrac{\dfrac{\textbf{stuck}}{\exists y P(x,y) \vdash \forall x P(x,y)}}{\exists y P(x,y) \vdash \exists y \forall x P(x,y)}\ \exists_R \text{ (with } y)}{\forall x \exists y P(x,y) \vdash \exists y \forall x P(x,y)}\ \forall_L \text{ (with } x)$$

The Excluded Middle

[Theorem] Drinker's Principle

In every bar, there is a person that, if s/he drinks, then everybody drinks.

- paradoxical ? let's prove it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D(t_0), D(x) \vdash D(x), \forall x D(x)}\text{ax}}{D(t_0) \vdash D(x), D(x) \Rightarrow \forall x D(x)}\Rightarrow_R}{D(t_0) \vdash D(x), \exists y(D(y) \Rightarrow \forall x D(x))}\exists_R \text{ (with } x\text{ !)}}{D(t_0) \vdash \forall x D(x), \exists y(D(y) \Rightarrow \forall x D(x))}\forall_R \text{ (}x\text{ fresh)}}{\vdash D(t_0) \Rightarrow \forall x D(x), \exists y(D(y) \Rightarrow \forall x D(x))}\Rightarrow_R}{\vdash \exists y(D(y) \Rightarrow \forall x D(x)), \exists y(D(y) \Rightarrow \forall x D(x))}\exists_R}{\vdash \exists y(D(y) \Rightarrow \forall x D(x))}\text{structural rule}$$

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- basically: either someone does not drink or everybody drinks.

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\begin{array}{ll}
\hline D(t_0), D(x) \vdash D(x), \forall x D(x) & \text{ax} \\
\hline D(t_0) \vdash D(x), D(x) \Rightarrow \forall x D(x) & \Rightarrow_R \\
\hline D(t_0) \vdash D(x), \exists y (D(y) \Rightarrow \forall x D(x)) & \exists_R \text{ (with } x \text{ !)} \\
\hline D(t_0) \vdash \forall x D(x), \exists y (D(y) \Rightarrow \forall x D(x)) & \forall_R \text{ (} x \text{ fresh)} \\
\hline \vdash D(t_0) \Rightarrow \forall x D(x), \exists y (D(y) \Rightarrow \forall x D(x)) & \Rightarrow_R \\
\hline \vdash \exists y (D(y) \Rightarrow \forall x D(x)), \exists y (D(y) \Rightarrow \forall x D(x)) & \exists_R \\
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  - ★ no constructive witness (the "best man")
  - ★ "Fermat's theorem is true" or not "Fermat's theorem is true"

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- basically: either someone does not drink or everybody drinks.
- not informative:
  - no constructive witness (the "best man")
  - "Fermat's theorem is true" or not "Fermat's theorem is true"
- PEM ($A \vee \neg A$ for free) rejected by Brouwer, Heyting, Kolmogorov (and all the constructivists).
  - bad also for the "proof-as-program" correpondence (Curry-Howard correspondence) until very recent advances (control operators)

The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

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$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\ \forall_R$$

The Intuitionistic Sequent Calculus (LJ)

$$\frac{}{\Gamma, A \vdash A}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{R2}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_R$$

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Example of Proof

- commutativity of the disjunction. Attempt #1:

$$A \vee B \vdash B \vee A$$

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$$\dfrac{A \vee B \vdash B}{A \vee B \vdash B \vee A}\ \vee_{R1}$$

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$$\dfrac{\dfrac{\dfrac{???}{A \vdash B} \quad \dfrac{}{B \vdash B}\,\text{ax}}{A \color{red}{\vee} B \vdash B}\,\vee_L}{A \vee B \vdash B \vee A}\,\vee_{R1}$$

Example of Proof

- commutativity of the disjunction. Attempt #2:

$$A \lor B \vdash B \lor A$$

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- commutativity of the disjunction. Attempt #2:

$$\frac{A \vee B \vdash A}{A \vee B \vdash B \vee A} \vee_{R2}$$

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}{A \lor B \vdash B \lor A}\,\lor_L
$$

Example of Proof

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$$\dfrac{\vee_{R2}\ \dfrac{\text{ax}\ \dfrac{}{A \vdash A}}{A \vdash B \color{red}{\vee} A} \qquad \dfrac{\dfrac{}{B \vdash B}\ \text{ax}}{B \vdash B \color{red}{\vee} A}\ \vee_{R1}}{A \vee B \vdash B \vee A}\ \vee_L$$

- compare with proofs in classical logic:

$$\vee_L\ \dfrac{\vee_R\ \dfrac{\text{ax}\ \dfrac{}{B \vdash B, A}}{B \vdash B \vee A} \qquad \dfrac{\dfrac{}{A \vdash B, A}\ \text{ax}}{A \vdash B \vee A}\ \vee_R}{A \vee B \vdash B \vee A} \qquad \dfrac{\dfrac{\text{ax}\ \dfrac{}{A \vdash B, A} \qquad \dfrac{}{B \vdash B, A}\ \text{ax}}{A \vee B \vdash B, A}\ \vee_L}{A \vee B \vdash B \vee A}\ \vee_R$$

- in particular, no *intuitionistic* proof of $\vdash A \color{red}{\vee} \neg A$: does it begins with $\vee_{R1}$, or with $\vee_{R2}$?

Weakening the statements

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- ★ Still controversial: *"If you are not innocent, then you are guilty"*
- ★ Exercises: Show, in classical logic, that $\vdash A \Rightarrow (\neg\neg A)$ and $\vdash (\neg\neg A) \Rightarrow A$.
  Harder: show $\vdash A \vee \neg A$ in intuitionistic logic + DN principle.

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  Harder: show $\vdash A \vee \neg A$ in intuitionistic logic + DN principle.

- from an intuitionistic point of view, $\neg\neg B$ is weaker than B:

$$\dfrac{}{A \vdash A}\ \text{ax}$$
$$\dfrac{A \vdash A}{A \vdash A \vee \neg A}\ \vee_{R1}$$
$$\dfrac{A \vdash A \vee \neg A}{\neg(A \vee \neg A), A \vdash}\ \neg_L$$
$$\dfrac{\neg(A \vee \neg A), A \vdash}{\neg(A \vee \neg A) \vdash \neg A}\ \neg_R$$
$$\dfrac{\neg(A \vee \neg A) \vdash \neg A}{\neg(A \vee \neg A) \vdash A \vee \neg A}\ \vee_{R2}$$
$$\dfrac{\neg(A \vee \neg A) \vdash A \vee \neg A}{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}\ \neg_L$$
$$\dfrac{\neg(A \vee \neg A), \neg(A \vee \neg A) \vdash}{\neg(A \vee \neg A) \vdash}\ \text{structural rule}$$
$$\dfrac{\neg(A \vee \neg A) \vdash}{\vdash \neg\neg(A \vee \neg A)}\ \neg_R$$

The principle of excluded-middle is not inconsistent

Double-Negation Translations

This drives us to try to systematically “weaken” classical formulæ to turn them into intuitionistically provable formulæ: Kolmogorov’s Translation

$$
\begin{array}{rcll}
P^{Ko} & = & \neg\neg P & \text{(atoms)} \\
(B \wedge C)^{Ko} & = & \neg\neg(B^{Ko} \wedge C^{Ko}) & \\
(B \vee C)^{Ko} & = & \neg\neg(B^{Ko} \vee C^{Ko}) & \\
(B \Rightarrow C)^{Ko} & = & \neg\neg(B^{Ko} \Rightarrow C^{Ko}) & \\
(\forall x A)^{Ko} & = & \neg\neg(\forall x A^{Ko}) & \\
(\exists x A)^{Ko} & = & \neg\neg(\exists x A^{Ko}) &
\end{array}
$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{Ko}, \lrcorner\Delta^{Ko} \vdash$ is provable in LJ.

Antinegation

$\lrcorner$ is an operator, such that:

- $\lrcorner\neg A = A$;
- $\lrcorner B = \neg B$ otherwise.

How does it work ?

Let us turn a (classical) proof of into a proof of its translation:

$$\dfrac{\dfrac{}{A \vdash A}\ \mathrm{ax}}{\vdash A \Rightarrow A}\ \Rightarrow_R \quad \longleftrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\ \mathrm{ax}}{\neg\neg A, \neg A \vdash}\ \neg L}{\neg\neg A \vdash \neg\neg A}\ \neg R}{\vdash (\neg\neg A) \Rightarrow (\neg\neg A)}\ \Rightarrow_R}{\neg((\neg\neg A) \Rightarrow (\neg\neg A)) \vdash}\ \neg L$$

Negation is bouncing:

- systematically: go from left to right, apply the same rule, and go from right to left

How does it work ?

Let us turn a (classical) proof of into a proof of its translation:

$$
\dfrac{\dfrac{}{A \vdash A}\,ax}{\vdash A \Rightarrow A}\,{\color{red}\Rightarrow_R}
\qquad \longleftrightarrow \qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{{\color{red}\neg A} \vdash {\color{red}\neg A}}\,ax}{{\color{red}\neg\neg A}, {\color{red}\neg A} \vdash}\,\neg L}{{\color{red}\neg\neg A} \vdash {\color{red}\neg\neg A}}\,\neg R}{\vdash ({\color{red}\neg\neg A}) \,{\color{red}\Rightarrow}\, ({\color{red}\neg\neg A})}\,{\color{red}\Rightarrow_R}}{{\color{red}\neg}(({\color{red}\neg\neg A}) \,{\color{red}\Rightarrow}\, ({\color{red}\neg\neg A})) \vdash}\,\neg L
$$

$$A \vdash A \longleftrightarrow \quad \vdash A \Rightarrow A \longleftrightarrow$$

Negation is bouncing:

- systematically: go from left to right, apply the same rule, and go from right to left
- many double negations are superflous: in the previous case, almost each of them (not hard to see that $\vdash A \Rightarrow A$ has an intuitionistic proof)

How does it work ?

Let us turn a (classical) proof of into a proof of its translation:

$$\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{\vdash A \Rightarrow A}\Rightarrow_R \qquad \longleftrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\,\text{ax}}{\neg\neg A, \neg A \vdash}\,\neg L}{\neg\neg A \vdash \neg\neg A}\,\neg R}{\vdash (\neg\neg A) \Rightarrow (\neg\neg A)}\Rightarrow_R}{\neg((\neg\neg A) \Rightarrow (\neg\neg A)) \vdash}\,\neg L$$

$$A \vdash A \longleftrightarrow \neg A \vdash \neg A \ldots \qquad \vdash A \Rightarrow A \longleftrightarrow \neg((\neg\neg A) \Rightarrow (\neg\neg A)) \vdash$$

Negation is bouncing:

- systematically: go from left to right, apply the same rule, and go from right to left
- many double negations are superflous: in the previous case, almost each of them (not hard to see that $\vdash A \Rightarrow A$ has an intuitionistic proof)
- Congratulations ! This is the topic of this talk

The Problem

Have the least possible $\neg\neg$ in the translated formula.

- what do we gain ? We preserve the strength of theorems.

Remarks on LK and LJ

- left-rules seem very similar in both cases
- so, lhs formulæ can be translated by themselves
- this accounts for polarizing the translations

Positive and Negative occurrences

- An occurrence of A in B is positive if:
  - $B = A$
  - B = $C \star D$ [$\star = \wedge, \vee$] and the occurrence of A is in C or in D and positive
  - B = $C \Rightarrow D$ and the occurrence of A is in C (resp. in D) and negative (resp. positive)
  - B = $\mathcal{Q}x\ C$ [$\mathcal{Q} = \forall, \exists$] and the occurrence of A is in C and is positive
- Dually for negative occurrences.

The Classical Sequent Calculus (LK)

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\ \forall_R$$

The Intuitionistic Sequent Calculus (LJ)

$$\frac{}{\Gamma, A \vdash A}\ \text{ax}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{R2}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x]}{\Gamma \vdash \forall x A}\ \forall_R$$

Light Kolmogorov's Translation

Moving negation from connectives to formulæ [DowekWerner]:

$$
\begin{array}{rcll}
B^K & = & B & \text{(atoms)} \\
(B \wedge C)^K & = & (\neg\neg B^K \wedge \neg\neg C^K) & \\
(B \vee C)^K & = & (\neg\neg B^K \vee \neg\neg C^K) & \\
(B \Rightarrow C)^K & = & (\neg\neg B^K \Rightarrow \neg\neg C^K) & \\
(\forall x A)^K & = & \forall x \neg\neg A^K & \\
(\exists x A)^K & = & \exists x \neg\neg A^K &
\end{array}
$$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^K, \neg\Delta^K \vdash$ is provable in LJ.

Correspondence

$A^{Ko} = \neg\neg A^K$

Polarizing Light Kolmogorov's translation

Warming-up. Consider left-hand and right-hand side formulæ:

LHS

$$
\begin{aligned}
B^K &= B \\
(B \wedge C)^K &= (\neg\neg B^K \wedge \neg\neg C^K) \\
(B \vee C)^K &= (\neg\neg B^K \vee \neg\neg C^K) \\
(B \Rightarrow C)^K &= (\neg\neg B^K \Rightarrow \neg\neg C^K) \\
(\forall x A)^K &= \forall x \neg\neg A^K \\
(\exists x A)^K &= \exists x \neg\neg A^K
\end{aligned}
$$

RHS

$$
\begin{aligned}
B^K &= B \\
(B \wedge C)^K &= (\neg\neg B^K \wedge \neg\neg C^K) \\
(B \vee C)^K &= (\neg\neg B^K \vee \neg\neg C^K) \\
(B \Rightarrow C)^K &= (\neg\neg B^K \Rightarrow \neg\neg C^K) \\
(\forall x A)^K &= \forall x \neg\neg A^K \\
(\exists x A)^K &= \exists x \neg\neg A^K
\end{aligned}
$$

Example of translation

$((A \vee B) \Rightarrow C)^K$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C$

$((A \vee B) \Rightarrow C)^K$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg C$

Polarizing Light Kolmogorov's Translation

Warming-up. Consider left-hand and right-hand side formulæ:

LHS

$$B^{K+} = B$$
$$(B \wedge C)^{K+} = (B^{K+} \wedge C^{K+})$$
$$(B \vee C)^{K+} = (B^{K+} \vee C^{K+})$$
$$(B \Rightarrow C)^{K+} = (\neg\neg B^{K-} \Rightarrow C^{K+})$$
$$(\forall x A)^{K+} = \forall x A^{K+}$$
$$(\exists x A)^{K+} = \exists x A^{K+}$$

RHS

$$B^{K-} = B$$
$$(B \wedge C)^{K-} = (\neg\neg B^{K-} \wedge \neg\neg C^{K-})$$
$$(B \vee C)^{K-} = (\neg\neg B^{K-} \vee \neg\neg C^{K-})$$
$$(B \Rightarrow C)^{K-} = (B^{K+} \Rightarrow \neg\neg C^{K-})$$
$$(\forall x A)^{K-} = \forall x \neg\neg A^{K-}$$
$$(\exists x A)^{K-} = \exists x \neg\neg A^{K-}$$

Example of translation

$((A \vee B) \Rightarrow C)^{K+}$ is $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow C$
$((A \vee B) \Rightarrow C)^{K-}$ is $(A \vee B) \Rightarrow \neg\neg C$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is still bouncing. Example:

$$\wedge_R \frac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\frac{\dfrac{\pi_1'}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash} \qquad \dfrac{\pi_2'}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash} \wedge_R$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is still bouncing. Example:

$$\wedge_R \;\dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \;\dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash} \qquad \dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}}\;\wedge_R}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is still bouncing. Example:

$$\wedge_R \; \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg_L \; \dfrac{\dfrac{\dfrac{\dfrac{\pi_1'}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}} \qquad \dfrac{\dfrac{\pi_2'}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}} \; \wedge_R}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is still bouncing. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \qquad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta}$$

is turned into:

$$\neg L \dfrac{\wedge_R \dfrac{\neg R \dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}} \qquad \neg R \dfrac{\dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}}}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}$$

Results on Polarized Kolmogorov's Translation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK, then $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ.

Proof: by induction. Negation is bouncing. Example:

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma^{K+}, \neg A^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-}}\neg_R \quad \dfrac{\dfrac{\pi'_2}{\Gamma^{K+}, \neg B^{K-}, \neg\Delta^{K-} \vdash}}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg B^{K-}}\neg_R}{\Gamma^{K+}, \neg\Delta^{K-} \vdash \neg\neg A^{K-} \wedge \neg\neg B^{K-}}\wedge_R}{\Gamma^{K+}, \neg(\neg\neg A^{K-} \wedge \neg\neg B^{K-}), \neg\Delta^{K-} \vdash}\neg_L$$

Theorem

If $\Gamma^{K+}, \neg\Delta^{K-} \vdash$ is provable in LJ, then $\Gamma \vdash \Delta$ is provable in LK.

Proof: ad-hoc generalization.

Gödel-Gentzen Translation

Disjunctions and existential quantifiers (the only problematic ones) are replaced by their De Morgan duals:

LHS			RHS		
B^{gg}	$=$	$\neg\neg B$	B^{gg}	$=$	$\neg\neg B$
$(A \wedge B)^{gg}$	$=$	$A^{gg} \wedge B^{gg}$	$(A \wedge B)^{gg}$	$=$	$A^{gg} \wedge B^{gg}$
$(A \vee B)^{gg}$	$=$	$\neg(\neg A^{gg} \wedge \neg B^{gg})$	$(A \vee B)^{gg}$	$=$	$\neg(\neg A^{gg} \wedge \neg B^{gg})$
$(A \Rightarrow B)^{gg}$	$=$	$A^{gg} \Rightarrow B^{gg}$	$(A \Rightarrow B)^{gg}$	$=$	$A^{gg} \Rightarrow B^{gg}$
$(\forall x A)^{gg}$	$=$	$\forall x A^{gg}$	$(\forall x A)^{gg}$	$=$	$\forall x A^{gg}$
$(\exists x A)^{gg}$	$=$	$\neg\forall x \neg A^{gg}$	$(\exists x A)^{gg}$	$=$	$\neg\forall x \neg A^{gg}$

Example of translation

$((A \vee B) \Rightarrow C)^{gg}$ is $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow \neg\neg C$

Theorem

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$ is provable in LJ.

Polarizing Gödel-Gentzen translation

Let us apply the same idea on this translation:

LHS			RHS		
B^p	$=$	B	B^n	$=$	$\neg\neg B$
$(B \wedge C)^p$	$=$	$B^p \wedge C^p$	$(B \wedge C)^n$	$=$	$B^n \wedge C^n$
$(B \vee C)^p$	$=$	$B^p \vee C^p$	$(B \vee C)^n$	$=$	$\neg(\neg B^n \wedge \neg C^n)$
$(B \Rightarrow C)^p$	$=$	$B^n \Rightarrow C^p$	$(B \Rightarrow C)^n$	$=$	$B^p \Rightarrow C^n$
$(\forall x B)^p$	$=$	$\forall x B^p$	$(\forall x B)^n$	$=$	$\forall x B^n$
$(\exists x B)^p$	$=$	$\exists x B^p$	$(\exists x B)^n$	$=$	$\neg\forall x \neg B^n$

Example of translation

$((A \vee B) \Rightarrow C)^p$ is $(\neg(\neg\neg\neg A \wedge \neg\neg\neg B)) \Rightarrow C$

$((A \vee B) \Rightarrow C)^n$ is $((A \vee B) \Rightarrow \neg\neg C$

Theorem ?

$\Gamma \vdash \Delta$ is provable in LK iff $\Gamma^{gg}, \lrcorner\Delta^{gg} \vdash$ is provable in LJ.

A Focus on LK $\rightarrow$ LJ

- less negations imposes more discipline. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi'_1}{\Gamma^p, \lrcorner A^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n} \quad ?? \dfrac{\dfrac{\pi'_2}{\Gamma^p, \lrcorner B^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash B^n}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner\Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.

A Focus on LK → LJ

- less negations imposes more discipline. Example:

$$\wedge_R \frac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \frac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi'_1}{\Gamma^p, \lrcorner A^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n} \quad ?? \dfrac{\dfrac{\pi'_2}{\Gamma^p, \lrcorner B^n, \lrcorner\Delta^n \vdash}}{\Gamma^p, \lrcorner\Delta^n \vdash B^n}}{\Gamma^p, \lrcorner\Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner\Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B).

A Focus on LK → LJ

- less negations imposes more discipline. Example:

$$\wedge_R \dfrac{\dfrac{\pi_1}{\Gamma \vdash A, \Delta} \quad \dfrac{\pi_2}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A \wedge B, \Delta} \quad \text{becomes} \quad \neg_L \dfrac{\wedge_R \dfrac{?? \dfrac{\dfrac{\pi'_1}{\Gamma^p, \lrcorner A^n, \lrcorner \Delta^n \vdash}}{\Gamma^p, \lrcorner \Delta^n \vdash A^n} \quad \dfrac{\dfrac{\pi'_2}{\Gamma^p, \lrcorner B^n, \lrcorner \Delta^n \vdash}}{\Gamma^p, \lrcorner \Delta^n \vdash B^n} ??}{\Gamma^p, \lrcorner \Delta^n \vdash A^n \wedge B^n}}{\Gamma^p, \neg(A^n \wedge B^n), \lrcorner \Delta^n \vdash}$$

- when A^n introduces negations ($\exists, \vee, \neg$ and atomic cases) ?? can be $\neg_R$ due to the behavior of $\lrcorner A^n$
- otherwise A^n remains of the rhs in the LJ proof.
- the next rule in π_1 and π_2 must be on A (resp. B).
- the liberty of sequent calculus is a sin! How to constrain it ?
- use Kleene's inversion lemma
- or ... this is exactly what focusing is about !

A Focused Classical Sequent Calculus

Sequent with focus

A focused sequent $\Gamma \vdash A ; \Delta$ has three parts:

- Γ and Δ
- A, the (possibly empty) stoup formula

$$\Gamma \vdash \underbrace{\quad . \quad}_{\text{stoup}} ; \Delta$$

- when the stoup is not empty, the next rule must apply on its formula,
- under some conditions, it is possible to move/remove a formula in/from the stoup.

A Focused Sequent Calculus

$$\frac{}{\Gamma, A \vdash\ .\ ;\ A, \Delta}\ ax$$

$$\frac{\Gamma, A, B \vdash\ .\ ;\ \Delta}{\Gamma, A \wedge B \vdash\ .\ ;\ \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash\ A\ ;\ \Delta \qquad \Gamma \vdash\ B\ ;\ \Delta}{\Gamma \vdash\ A \wedge B\ ;\ \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash\ .\ ;\ \Delta \qquad \Gamma, B \vdash\ .\ ;\ \Delta}{\Gamma, A \vee B \vdash\ .\ ;\ \Delta}\ \vee_L \qquad \frac{\Gamma \vdash\ .\ ;\ A, B, \Delta}{\Gamma \vdash\ .\ ;\ A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash\ A\ ;\ \Delta \qquad \Gamma, B \vdash\ .\ ;\ \Delta}{\Gamma, A \Rightarrow B \vdash\ .\ ;\ \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash\ B\ ;\ \Delta}{\Gamma \vdash\ A \Rightarrow B\ ;\ \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash\ .\ ;\ \Delta}{\Gamma, \exists x A \vdash\ .\ ;\ \Delta}\ \exists_L \qquad \frac{\Gamma \vdash\ .\ ;\ A[t/x], \Delta}{\Gamma \vdash\ .\ ;\ \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash\ .\ ;\ \Delta}{\Gamma, \forall x A \vdash\ .\ ;\ \Delta}\ \forall_L \qquad \frac{\Gamma \vdash\ A[c/x]\ ;\ \Delta}{\Gamma \vdash\ \forall x A\ ;\ \Delta}\ \forall_R$$

$$\frac{\Gamma \vdash\ A\ ;\ \Delta}{\Gamma \vdash\ .\ ;\ A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash\ .\ ;\ A, \Delta}{\Gamma \vdash\ A\ ;\ \Delta}\ \text{release}$$

A Focused Sequent Calculus

$$\frac{\Gamma \vdash A \,;\, \Delta}{\Gamma \vdash \,.\,;\, A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \,.\,;\, A, \Delta}{\Gamma \vdash A \,;\, \Delta}\ \text{release}$$

Characteristics:

- in release, A is either atomic or of the form $\exists x B, B \vee C$ or $\neg B$;
- in focus, the converse holds: A must not be atomic, nor of the form $\exists x B, B \vee C$ nor $\neg B$.
- the *synchronous* (outside the stoup) right-rules are $\exists_R$, $\neg_R$, $\vee_R$ and (atomic) axiom: the exact places where $\{.\}^n$ introduces negation

Theorem

If $\Gamma \vdash \Delta$ is provable in LK then $\Gamma \vdash .; \Delta$ is provable.

Proof: use Kleene's inversion lemma (holds for all connectives/quantifiers, except $\exists_R$ and $\forall_L$).

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A \,;\, \Delta}{\Gamma \vdash \,.\, ; A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash \,.\, ; A, \Delta}{\Gamma \vdash A \,;\, \Delta}\ \text{release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^p, \lrcorner\Delta^n \vdash A^n$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule

Translating Focused Proofs in LJ

$$\frac{\Gamma \vdash A\,;\,\Delta}{\Gamma \vdash .\,;\,A, \Delta}\ \text{focus} \qquad \frac{\Gamma \vdash .\,;\,A, \Delta}{\Gamma \vdash A\,;\,\Delta}\ \text{release}$$

Theorem

If $\Gamma \vdash A; \Delta$ in focused LK, then $\Gamma^p, \lrcorner\Delta^n \vdash A^n$ in LJ

- release is translated by the $\neg_R$ rule
- focus is translated by the $\neg_L$ rule
- $\lrcorner\Delta^n$ removes the trailing negation on $\exists^n$ $(\neg\forall\neg)$, $\vee^n$ $(\neg \wedge \neg)$, $\neg^n$ $(\neg)$ and atoms $(\neg\neg)$
- what a surprise: focus is forbidden on them, so rule on the lhs:

LK rule	$\exists_R$	$\neg_R$	$\vee_R$	ax.
LJ rule	$\forall_L$	nop	$\wedge_L$	$\neg_L$ + ax.

Going further: Kuroda's translation

Originating from Glivenko's remark for propositional logic:

Theorem [Glivenko]

if $\vdash A$ in LK, then $\vdash \neg\neg A$ in LJ.

Kuroda's $\neg\neg$-translation:

$$
\begin{array}{rcll}
B^{Ku} & = & B & \text{(atoms)} \\
(B \wedge C)^{Ku} & = & B^{Ku} \wedge C^{Ku} & \\
(B \vee C)^{Ku} & = & B^{Ku} \vee C^{Ku} & \\
(B \Rightarrow C)^{Ku} & = & B^{Ku} \Rightarrow C^{Ku} & \\
(\forall x A)^{Ku} & = & \forall x \neg\neg A^{Ku} & \\
(\exists x A)^{Ku} & = & \exists x A^{Ku} &
\end{array}
$$

Theorem [Kuroda]

$\Gamma \vdash \Delta$ in LK iff $\Gamma^{Ku}, \neg\Delta^{Ku} \vdash$ in LJ.

- restarts double-negation everytime we pass a universal quantifier.

Combining Kuroda's and Gentzen-Gödel's translations

- work of Frédéric Gilbert (2013), who noticed:
  1. Kuroda's translation of $\forall x \forall y A$

     $$\forall x \neg\neg \forall y \neg\neg A \text{ can be simplified: } \forall x \forall y \neg\neg A$$

  2. $\neg\neg A$ itself can be treated *à la* Gentzen-Gödel
  3. and of course with polarization

Reminder:

Gödel-Gentzen

$$\begin{aligned}
\varphi(P) &= \neg\neg P \\
\varphi(A \wedge B) &= \varphi(A) \wedge \varphi(B) \\
\varphi(A \vee B) &= \neg\neg(\varphi(A) \vee \varphi(B)) \\
\varphi(A \Rightarrow B) &= \varphi(A) \Rightarrow \varphi(B) \\
\varphi(\exists x A) &= \neg\neg \exists x \varphi(A) \\
\varphi(\forall x A) &= \forall x \varphi(A)
\end{aligned}$$

Kuroda

$$\begin{aligned}
\psi(P) &= P \\
\psi(A \wedge B) &= \psi(A) \wedge \psi(B) \\
\psi(A \vee B) &= \psi(A) \vee \psi(B) \\
\psi(A \Rightarrow B) &= \psi(A) \Rightarrow \psi(B) \\
\psi(\exists x A) &= \exists x \psi(A) \\
\psi(\forall x A) &= \forall x \neg\neg \psi(A)
\end{aligned}$$

Combining Kuroda's and Gentzen-Gödel's translations

- How does it work ?

GG

$$\begin{aligned}
\varphi(P) &= \neg\neg P \\
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Kuroda

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\psi(A \wedge B) &= \psi(A) \wedge \psi(B) \\
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\psi(A \Rightarrow B) &= \psi(A) \Rightarrow \psi(B) \\
\psi(\exists x A) &= \exists x \psi(A) \\
\psi(\forall x A) &= \forall x \neg\neg\psi(A)
\end{aligned}$$

Combining Kuroda's and Gentzen-Gödel's translations

- How does it work ?

RHS	LHS	Kuroda
$\varphi(P) = \neg\neg P$	$\chi(P) = P$	$\psi(P) = P$
$\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$	$\chi(A \wedge B) = \chi(A) \wedge \chi(B)$	$\psi(A \wedge B) = \psi(A) \wedge \psi(B)$
$\varphi(A \vee B) = \neg\neg\psi(A) \vee \psi(B)$	$\chi(A \vee B) = \chi(A) \vee \chi(B)$	$\psi(A \vee B) = \psi(A) \vee \psi(B)$
$\varphi(A \Rightarrow B) = \chi(A) \Rightarrow \varphi(B)$	$\chi(A \Rightarrow B) = \psi(A) \Rightarrow \chi(B)$	$\psi(A \Rightarrow B) = \chi(A) \Rightarrow \psi(B)$
$\varphi(\exists x A) = \neg\neg\exists x \psi(A)$	$\chi(\exists x A) = \exists x \chi(A)$	$\psi(\exists x A) = \exists x \psi(A)$
$\varphi(\forall x A) = \forall x \varphi(A)$	$\chi(\forall x A) = \forall x \chi(A)$	$\psi(\forall x A) = \forall x \varphi(A)$

- How to prove that ? Refine focusing into phases.

Example of translation

$\chi((A \vee B) \Rightarrow C)$ is $(A \vee B) \Rightarrow C$

$\varphi((A \vee B) \Rightarrow C)$ is $(A \vee B) \Rightarrow \neg\neg C$

$$\frac{}{\Gamma, A \vdash \ . ; A, \Delta} \text{ax}$$

$$\frac{\Gamma, A, B \vdash \ . ; \Delta}{\Gamma, A \wedge B \vdash \ . ; \Delta} \wedge_L \qquad \frac{\Gamma \vdash A ; \Delta \qquad \Gamma \vdash B ; \Delta}{\Gamma \vdash A \wedge B ; \Delta} \wedge_R$$

$$\frac{\Gamma, A \vdash \ . ; \Delta \qquad \Gamma, B \vdash \ . ; \Delta}{\Gamma, A \vee B \vdash \ . ; \Delta} \vee_L \qquad \frac{\Gamma \vdash \ . ; A, B, \Delta}{\Gamma \vdash \ . ; A \vee B, \Delta} \vee_R$$

$$\frac{\Gamma \vdash A ; \Delta \qquad \Gamma, B \vdash \ . ; \Delta}{\Gamma, A \Rightarrow B \vdash \ . ; \Delta} \Rightarrow_L \qquad \frac{\Gamma, A \vdash B ; \Delta}{\Gamma \vdash A \Rightarrow B ; \Delta} \Rightarrow_R$$

$$\frac{\Gamma, A[c/x] \vdash \ . ; \Delta}{\Gamma, \exists x A \vdash \ . ; \Delta} \exists_L \qquad \frac{\Gamma \vdash \ . ; A[t/x], \Delta}{\Gamma \vdash \ . ; \exists x A, \Delta} \exists_R$$

$$\frac{\Gamma, A[t/x] \vdash \ . ; \Delta}{\Gamma, \forall x A \vdash \ . ; \Delta} \forall_L \qquad \frac{\Gamma \vdash A[c/x] ; \Delta}{\Gamma \vdash \forall x A ; \Delta} \forall_R$$

$$\frac{\Gamma \vdash A ; \Delta}{\Gamma \vdash \ . ; A, \Delta} \text{focus} \qquad \frac{\Gamma \vdash \ . ; A, \Delta}{\Gamma \vdash A ; \Delta} \text{release}$$

Results

Theorem [Gilbert]

if $\Gamma_0, \neg\Gamma_1 \vdash A; \Delta$ in $LK_{\uparrow\downarrow}$ then $\chi(\Gamma_0), \neg\psi(\Gamma_1), \neg\psi(\Delta) \vdash \varphi(A)$ in LJ.

Theorem [Gilbert]

$A \mapsto \varphi(A)$ is minimal among the $\neg\neg$-translations.

- 58% of Zenon's modulo proofs are secretly constructive
- polarizing the translation of rewrite rules in Deduction modulo:
  - problem with cut elimination: a rule is usable in the lhs and rhs
  - back to a non-polarized one
  - further work: use polarized Deduction modulo
- further work: polarize Krivine's translation

What you hopefully should remember:

- Focusing is a perfect tool to remove double-negations;
- antinegation $\lrcorner$.