A note and a correction to *Completeness and Cut Elimination in the Intuitionistic Theory of Types*

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1 Introduction

This paper aims to modify an erroneous construction of [3].

2 The problem and its correction

A key technical lemma for proving completeness in [3] (Theorem 5.9, on page 842) relies on a tableau-style definition of **consistency** (definition 5.5) of a Hintikka set of signed forcing statements in type theory.

For the proof of the lemma to work, one must strengthen this definition, and use, in place of $\mathsf{F}_p(\mathcal{H})$, the set of negative forcing statements at world p, *its closure* $\bigvee \mathsf{F}_p(\mathcal{H})$ *under finite disjunctions.* The correct definition should read as follows, with the key changes in boldface:

Definition 2.1 (Def. 5.5, of [3] corrected) *A K-Hintikka set (resp. Hintikka path) is ICTT-consistent if for any p, and any* **finite disjunction** *B of formulas in $\mathsf{F}_p(\mathcal{H})$,*

$$\mathsf{T}_p(\mathcal{H}) \not\vdash B,$$

where provability means in the ICTT sequent calculus.

The change is required to ensure the correctness of one of the disjunction cases of the induction proof of Theorem 5.9, on page 842, which states that if a theory Γ fails to prove some formula A there is a consistent K-Hintikka path π for (Γ, A). The proof proceeds by *showing that any finite consistent path in a partially developed tableau must have a consistent extension*, using the tableau construction of Lemma 5.8, p. 839. One must examine every case ($\mathsf{T}p \Vdash C$ or $\mathsf{F}p \Vdash C$ for all possible formulas C) for the least unused node ν on π, and show that if all path extensions π' corresponding to that case are inconsistent, then so is π.

In the $\mathsf{T}p \Vdash \vee BC$ case the tableau rules generate an extension of π into *two* paths, one labeled by $\mathsf{T}p \Vdash B$, the other by $\mathsf{T}p \Vdash C$. If both paths so generated are now inconsistent, then for some formulas D, D' in $\mathsf{F}_p(\pi)$ we have $\mathsf{T}_p(\pi), B \vdash D$ and $\mathsf{T}_p(\pi), C \vdash D'$, from which, by $\vee$-left and $\vee$-right, $\mathsf{T}_p(\pi), B \vee C \vdash D \vee D'$. Since $B \vee C \in \mathsf{T}_p(\pi)$ this yields $\mathsf{T}_p(\pi) \vdash D \vee D'$. This does *not* contradict the consistency of π unless we use the second definition of consistency above, in which we take the disjunctive closure of $\mathsf{F}_p(\pi)$. With the new definition we have an immediate contradiction, but now the other cases must be shown to work as well. This calls for the application of certain technical lemmas about the associativity, and commutativity of of $\vee$ and the legitimacy of derived rules in the presence of disjunction (and the absence of the cut rule), many of which were established (in roughly equivalent form) in [1], and in [2]. Since the formal system used here is not exactly the same as those used in the references cited, we will sketch several of the proofs below.

Lemma 2.2 (Associativity) *Let Γ be a set of formulae and A, B, C be fomulae, let $P \equiv_\lambda A \vee (B \vee C)$ or $P \equiv_\lambda (B \vee C)$. From a cut-free proof of $\Gamma \vdash P$ we can construct a cut-free proof of $\Gamma \vdash (A \vee B) \vee C$.*

Proof. We appeal to the fact that in a cut-free proof, the formula P must have been the result of prior application(s) of the right $\vee$ rule.

More formally, we show by induction on the depth of the given proof tree, that from a cut-free proof ϑ of $\Gamma \vdash A \vee (B \vee C)$ **or** a proof ϑ' of $\Gamma \vdash B \vee C$ we can yield the desired cut-free proof. We consider some of the cases involved.

We will assume that the *axiom* rule allowing free inference of $\Gamma, A \vdash A$ is only allowed for atomic formulas A. This variant of ICTT is trivially equivalent to the formulation in the paper, and it makes the base case of the proof of this and subsequent lemmas vacuously true.

First, consider the proof ϑ and suppose the last rule used is a left rule. This can take one of three forms, shown below. We consider the first two, the first corresponding to the $\wedge, \exists, \forall$ cases, the second corresponding to the $\vee$ case.

$$\frac{\begin{array}{c}\vdots\vartheta_1\\ \Gamma' \vdash A \vee (B \vee C)\end{array}}{\Gamma \vdash A \vee (B \vee C)}\,\ell_1 \qquad \frac{\begin{array}{cc}\vdots\vartheta_1 & \vdots\vartheta_2\\ \Gamma' \vdash A \vee (B \vee C) & \Gamma'' \vdash A \vee (B \vee C)\end{array}}{\Gamma \vdash A \vee (B \vee C)}\,\vee_\ell$$

$$\frac{\begin{array}{cc}\vdots\vartheta_1 & \vdots\vartheta_2\\ \Gamma', K_1 \vdash A \vee (B \vee C) & \Gamma'' \vdash K_2\end{array}}{\Gamma \vdash A \vee (B \vee C)}\,\supset_\ell$$

In the first case, apply the induction hypothesis to the shorter proof ϑ_1 to yield a proof of $\Gamma' \vdash (A \vee B) \vee C$, and then use the corresponding ℓ_1 rule to obtain $\Gamma \vdash (A \vee B) \vee C$. In the second case apply the induction hypothesis to both proofs ϑ_1 and ϑ_2, and then use the $\vee_\ell$ rule to obtain the desired conclusion. The third form a left rule can take is $\supset_\ell$. This case is treated similarly, applying the induction hypothesis to ϑ_1, and then the $\supset_\ell$ rule.

If the last rule used in ϑ was a *right rule* it must be the $\vee$ rule, since the proof is cut-free. If the rule used was

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee (B \vee C)}$$

then from $\Gamma \vdash A$ we infer $\Gamma \vdash A \vee B$ and then $\Gamma \vdash (A \vee B) \vee C$. If the last rule used was a right $\vee$ applied to $B \vee C$ we use the induction hypothesis on proofs of $B \vee C$ to obtain the result. Now suppose $P \equiv_\lambda (B \vee C)$ is derived by the cut-free proof ϑ'. We proceed as above, considering the cases for the last rule used in the proof. The left-rule cases use the induction hypothesis in the same way. If the last rule was a right rule, it was $\vee$-right applied to B or C and the result is immediate. □

Using similar arguments we obtain also the derivability of contraction, and commutativity of disjuncts, as well as the following lemma, similar in spirit to the formal system $\mathcal{G}$ introduced in section 4.2 of [1], and subsequently shown equivalent to Church's (classical) theory of types. Proofs can also be found, for a related formal system in [2].

Lemma 2.3 *Let A,B,C, P, Q, R be formulae and Γ_1, Γ_2 be sets of formulae.*

1. *If P is $A \vee C$ or A, then from cut-free proofs of $\Gamma_1 \vdash P$ and $\Gamma_2 \vdash B \vee C$ we can construct a cut-free proof of $\Gamma_1, \Gamma_2 \vdash (A \wedge B) \vee C$.*

2. *from a cut-free proof of* $\Gamma \vdash Bt \vee C$ *we can construct a cut-free proof of* $\Gamma \vdash (\Sigma B) \vee C$.

3. *from cut-free proofs of* $\Gamma_1 \vdash A \vee C$ *and* $\Gamma_2, B \vdash C$ *we can construct a cut-free proof of* $\Gamma_1, \Gamma_2, A \supset B \vdash C$

Lem. 2.3 above has indeed strong links with the syntactical soundness of multi-succedent intuitionistic sequent calculus w.r.t. the usual intuitionistic sequent calculus, as shown in [2]. The point is that multiple conclusions are not harmful as long as we do not have to apply $\supset_R$ or $\forall_R$ rules. See e.g. [4].

Now we are in a position to establish the cited consistency theorem.

Theorem 2.4 (Th. 5.9 [3], corrected) *Suppose* $\Gamma \not\vdash A$. *Then there is a consistent* K*-Hintikka path* π *for* (Γ, A).

Proof. We show that if π is a consistent finite path in some partially developed tableau τ_n for (Γ, A) then at least one of the ways it is extended at some stage preserves consistency.

Suppose π is a path through ν, the least unused node on π with entry $\mathsf{B}p \Vdash B$. We consider only the cases undergoing a significant change with respect to [3]:

$\mathsf{T}p \Vdash \wedge BC$: almost the same. Replace D in $\mathsf{F}_p(\pi)$ by D in $\bigvee \mathsf{F}_p(\pi)$.

$\mathsf{F}p \Vdash \wedge BC$: π is extended to two new paths π_0, π_1. Observe that $\mathsf{T}_p(\pi) = \mathsf{T}_p(\pi_0) = \mathsf{T}_p(\pi_1)$, and that $\mathsf{F}_p(\pi_0) = \mathsf{F}_p(\pi) \cup \{B\}$ and $\mathsf{F}_p(\pi_1) = \mathsf{F}_p(\pi) \cup \{C\}$. If both extensions are inconsistent, the only possible violations of consistency are:

$$\mathsf{T}_p(\pi) \vdash D \text{ and } \mathsf{T}_p(\pi) \vdash D'$$

for D in $\bigvee \mathsf{F}_p(\pi_0)$ and D' in $\bigvee \mathsf{F}_p(\pi_1)$. Using associativity and contraction for disjunctive formulas (see remarks after Lemma 2.2) we can replace D by $B \vee E$ and D' by $C \vee E'$ for $E, E' \in \bigvee \mathsf{F}_p(\pi)$. Letting $E'' = E \vee E'$ and using $\vee$-right, we have,

$$\mathsf{T}_p(\pi) \vdash B \vee E'' \text{ and } \mathsf{T}_p(\pi) \vdash C \vee E''$$

We use Lem. 2.3 to get a proof of $\mathsf{T}_p(\pi) \vdash (B \wedge C) \vee E''$. Since $B \wedge C \in \mathsf{F}_p(\pi)$, we have $(B \wedge C) \vee E'' \in \bigvee \mathsf{F}_p(\pi)$ for a contradiction.

$\mathsf{T}p \Vdash \vee BC$ is the same as the previous case, but we do not have to use Lem. 2.3 since $\mathsf{F}_p(\pi)$ does not change.

$\mathsf{F}p \Vdash \vee BC$: π extends into one new path, adding the statements $\mathsf{F}p \Vdash B, \mathsf{F}p \Vdash C$. If the new path is inconsistent, it could only be because there is a proof of:

$$\mathsf{T}_p(\pi) \vdash (B \vee C) \vee E$$

for some E in $\bigvee \mathsf{F}_p(\pi)$, making use of contraction, associativity and commutativity of $\vee$ and $\vee$-right as needed. But $B \vee C \in \mathsf{F}_p(\pi)$ and hence $(B \vee C) \vee E$ is in $\bigvee \mathsf{F}_p(\pi)$, so π itself must be inconsistent.

$\mathsf{T}p \Vdash B \supset C$: Let $q \geq p$ the world introduced in the path extension. If both new paths π_0 and π_1 are inconsistent, then it can only be because we have the following proofs:

$$\mathsf{T}_q(\pi) \vdash B \vee E \text{ and } \mathsf{T}_q(\pi), C \vdash E'$$

where E and E', and hence their disjunction $E'' = E \vee E'$ is in $\bigvee \mathsf{F}_q(\pi)$.

This follows from the fact $\mathsf{F}_q(\pi_0) = \{B\} \cup \mathsf{F}_q(\pi)$, $\mathsf{F}_q(\pi_1) = \mathsf{F}_q(\pi)$ and $\mathsf{T}_q(\pi_0) = \mathsf{T}_q(\pi)$, $\mathsf{T}_q(\pi_1) = \{C\} \cup \mathsf{T}_q(\pi)$. Now (by $\vee$-right) we may replace E and E' above by E'', then apply Lem. 2.3 and get a proof of: $\mathsf{T}_q(\pi) \vdash E''$, showing there is a contradiction already present on the original path.

$\mathsf{F}p \Vdash B \supset C$: if the new path is inconsistent, this only can occur at the new world p' introduced. Since it is new, we have $\mathsf{F}_{p'}(\pi') = \{C\}$ and $\mathsf{T}_{p'}(\pi') = \{B\} \cup \mathsf{T}_p(\pi)$. Hence, we have a proof of:

$$\mathsf{T}_p(\pi), B \vdash C$$

and we can apply the $\supset$ right rule to obtain the inconsistency of π.

$\mathsf{T}p \Vdash \Sigma_{o(o\alpha)}B$ is identical to the corresponding case of [3], replacing $D \in \mathsf{F}_p(\pi)$ by $D \in \bigvee \mathsf{F}_p(\pi)$.

$\mathsf{F}p \Vdash \Sigma_{o(o\alpha)}B$: the path π is extended with the two nodes $\{\mathsf{F}p \Vdash \Sigma_{o(o\alpha)}B, \mathsf{F}p \Vdash BC\}$ for the appropriate witness C. If the new path is inconsistent, we have, by use of contraction, commutativity and associativity of disjunction, a proof of:

$$\mathsf{T}_p(\pi) \vdash (BC) \vee E$$

for some E in $\bigvee \mathsf{F}_p(\pi)$. Applying Lem. 2.3, we can find a proof of $\mathsf{T}_p(\pi) \vdash \Sigma_{o(o\alpha)}B \vee E$. But $\Sigma_{o(o\alpha)}B \vee E$ is in $\bigvee \mathsf{F}_p(\pi)$, contradicting the consistency of π.

$\mathsf{T}p \Vdash \Pi_{o(o\alpha)}B$: the path π is extended with the two nodes $\{\mathsf{T}q \Vdash \Pi_{o(o\alpha)}B, \mathsf{T}q \Vdash BC\}$ for the appropriate $q \geq p$ and C. If the new path is inconsistent, it must be because we have a proof of:

$$\mathsf{T}_q(\pi), (BC) \vdash E$$

with $E \in \bigvee \mathsf{F}_q(\pi)$, since $\mathsf{T}_q(\pi') = \mathsf{T}_q(\pi) \cup \{\Pi_{o(o\alpha)}B, BC\} = \mathsf{T}_q(\pi) \cup \{BC\}$ and $\mathsf{F}_q(\pi') = \mathsf{F}_q(\pi)$. We apply $\forall_L$ rule to obtain the inconsistency of π, yielding a contradiction.

$\mathsf{F}p \Vdash \Pi_{o(o\alpha)}B$: The path π is extended to a new path π' with the single entry $\mathsf{F}p' \Vdash Bc_\alpha$ for the appropriate *new* $p' \geq p$ and *fresh* constant c_α. If π is consistent and π' fails to be, since $\mathsf{T}_{p'}(\pi') = \mathsf{T}_p(\pi)$ and $\mathsf{F}_{p'} = \{Bc_\alpha\}$ (p' is new) we must have:

$$\mathsf{T}_p(\pi) \vdash Bc_\alpha$$

Generalizing on the fresh constant, we obtain, by $\forall_R$,

$$\mathsf{T}_p(\pi) \vdash \Pi_{o(o\alpha)}B$$

yielding the inconsistency of π for a contradiction, which completes the proof.

□

References

[1] Peter B Andrews. Resolution in type theory. *The Journal of Symbolic Logic*, 36(3):414–432, September 1971.

[2] Richard Bonichon and Olivier Hermant. On constructive cut admissibility in deduction modulo. 2007. To be published in TYPES 2006 Post-Proceedings.

[3] Mary De Marco and James Lipton. Completeness and cut elimination in Church's intuitionistic theory of types. *Journal of Logic and Computation*, 15(6):821–854, December 2005.

[4] James Lipton and Michael J. O'Donnell. Some intuitions behind realizability semantics for constructive logic: Tableaux and lauchli countermodels. *Annals of Pure and Applied Logic*, 81(1-3):187–239, 1996.

APPENDIX

Proof. [of Lemma 2.3] We recall that, as mentioned in [3], (page 825) weakening on the left is a derived rule in ICTT, and that in our sequents, antecedents are sets (hence contraction and exchange on the left is taken for granted).

Now, for all three conclusions, we induct on the depth of the cut-free proof ϑ_1 of the first sequent, $\Gamma_1 \vdash P$ in 2.3.1, $\Gamma \vdash Bt \vee C$ in 2.3.2 and $\Gamma_1 \vdash A \vee C$ in 2.3.3.

If the last rule used in ϑ_1 is a left rule, the proof proceeds as in Lemma 2.2. We consider the three types of left rules, applying the induction hypothesis to the antecedent(s) and then the left rule in question.

If the last rule used is a *right rule*, we consider the three claims of the lemma, separately:

2.3.1:

If P is $A \vee C$ then the rule must have been $\vee_r$. If it was applied to introduce the left disjunct:

$$\frac{\Gamma_1 \vdash C}{\Gamma_1 \vdash A \vee C}$$

then we can obtain the desired proof from $\Gamma_1 \vdash C$ just by one use of $\vee_r$ and weakening on the left. If applied to introduce the right disjunct

$$\frac{\begin{array}{c}\vdots \vartheta_1' \\ \Gamma_1 \vdash A\end{array}}{\Gamma_1 \vdash A \vee C}$$

then apply the induction hypothesis (for the case $P = A$) to the shorter proof ϑ_1'.

If P is A, we must induct on the depth of the other given proof (of $\Gamma_2 \vdash B \vee C$). The arguments are similar to the ones just used. We permute the left rules with the inductive hypothesis, and if the last rule used was a right rule, we have either a proof of $\Gamma_2 \vdash B$, and using $\wedge_r$ and then $\vee_r$ we have the desired conclusion, or a proof of $\Gamma_2 \vdash C$ and one use of $\vee_r$ and weakening on the left will suffice.

2.3.2:

If the last rule used was a right rule, necessarily $\vee_r$, applied to $\Gamma \vdash Bt$, then use $\exists_r$ to get $\Gamma \vdash \Sigma B$, and then $\vee_r$ to get $\Gamma \vdash \Sigma B \vee C$. If the premise was $\Gamma \vdash C$, we obtain the desired conclusion with a simple $\vee_r$.

2.3.3:

If the last rule used in the proof ϑ_1 of $\Gamma \vdash A \vee C$ was a right rule, then it was either $\vee_r$ applied to $\Gamma_1 \vdash C$, and, just by weakening on the left, we obtain $\Gamma_1, \Gamma_2, A \supset B \vdash C$, or it was $\vee_r$ applied to $\Gamma_1 \vdash A$. Then we can apply $\supset_\ell$ directly to $\Gamma_1 \vdash A$ and $\Gamma_2, B \vdash C$ to obtain the desired conclusion.

The base case of our induction, namely that the main premise involved was an instance of the *axiom* rule $\Gamma, A \vdash A$ is immediate, since, in this note, we only allow this rule to be applied to atomic formulas A (see the note at the beginning of the proof of associativity, above).

□