Cut Elimination in the Intuitionistic Theory of Types with Axioms and Rewriting Cuts, Constructively

Olivier Hermant, James Lipton

1 Introduction

We will give a constructive proof for a semantic cut elimination theorem for Intuitionistic Church's Theory of Types (ICTT) extended with certain axioms. The argument extends techniques of Prawitz, Takahashi and Andrews, as well as those used in [5]. To the authors' knowledge it is the first constructive semantic proof of cut elimination for ICTT, and the extensions considered.

We recall that the central problem in proving cut-elimination for certain *impredicative* higher-order logics is that Gentzen's approach, based on an induction on a measure that combines proof-depth and formula complexity, does not work because the natural subformula ordering that places instances $M[t/x]$ below quantified formulae such as $\exists x.M$ is not a well-ordering. Such instances can be more complex, as can be seen by taking $M = \exists x.x$ and taking $t = M \wedge A$ for any A, for example. This problem, originally known as the Takeuti conjecture (the claim that second-order logic admits cut-elimination, 1953), was solved positively, and non-constructively, independently by Tait (1966), Takahashi(1967), Prawitz and others using semantic means, and constructively via a strong normalization proof, in 1971 by Girard[10]. In 1970 Andrews[1] gave a non constructive proof along the lines of Takahashi's V-complex construction for Church's classical theory of types. Dragalin [8] showed how to give a constructive semantic proof for higher-order classical logic. The second author gave a semantic proof for an intuitionistic formulation of Church's type theory in [5], also non-constructive.

The proof makes use of the following components. We define a class of models, a type-theoretic version of Scott-Fourman Ω sets, and show completeness constructively for the cut-free fragment of a number of type theories discussed in this paper. This gives cut-admissibility of those fragments as an immediate corollary. The impredicativity of the formal system involved makes it impossible to define a semantics along conventional lines, in the absence, a priori, of cut, or to prove completeness. The problem is that one cannot use induction on the subformula order to define truth, or use transitivity of entailment. As a result, as in the semantic proofs cited, in particular Takahashi's and Andrews', one must start from a tableau style construction of a partial model, called a *semivaluation*, and extend to a full model in a non-deterministic fashion, by assigning candidate truth values to formulae, then using induction on types to make the construction work. In [5] a series of algebraic conditions were given for partial truth assignments which guarantee that they can be extended to models. These conditions are applied here to a new, syntactic notion of semivaluation based on mapping formulae A to sets generated by contexts Γ for which cut-free proofs of $\Gamma \vdash A$ exist, inspired by results of Okada and the first author. This yields a constructive proof for the theory of types. This argument is then extended to include various types of sequent axioms which encapsulate rewriting rules for formulae.

The idea of building rewriting into logic is inspired in a formal system that combines sequent proofs, higher-order equational constraints and term rewriting, called Deduction Modulo, invented by Dowek, Hardin, Kirchner and Werner [6, 7]. The aim of such a formal system is to integrate computation directly into logic in a new way. Cut elimination for various fragments of this system, which does not, in general, satisfy strong normalization, has been studied by Hermant and Dowek.

1.1 Outline

In a first part, we define the semantic space we are working in, and the tools we need to prove this theorem, along the lines of [5]. The main novelty is the definition of the semi-valuations used, that makes the proof constructive.

In a second part, we show that the argument works for an extension of ICTT with non-logical axioms (as for instance $\vdash \forall P.P \vee \neg P$, which gives us back the classical version of Church's Theory of Types), under the proviso that we allow specific new sequent rules which we will call *axiomatic cuts*.

We investigate this line further, and show that if we constrain the non-logical axioms to have a specific form (they have to satisfy a so-called *sign-preserving condition*) one can restrict the *axiomatic cuts* to take the form of *rewrite cuts*, where rewriting rules are expressed as cut-style rules of inference.

2 The Formal System: a sketch

For definitions of types, terms and reduction in the intuitionistic formulation of Church's Theory of types, due originally to Miller et al. [16], we refer the

$$\frac{}{\Gamma \vdash \top} \qquad \frac{}{\Gamma, U \vdash U} \qquad \frac{}{\Gamma, \bot \vdash \bot}$$

$$\frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A} \wedge_L \qquad \frac{\Gamma \vdash B \quad \Gamma \vdash C}{\Gamma \vdash B \wedge C} \wedge_R$$

$$\frac{\Gamma, B \vdash A \quad \Gamma, C \vdash A}{\Gamma, B \vee C \vdash A} \vee_L \qquad \frac{\Gamma \vdash B_i}{\Gamma \vdash B_1 \vee B_2} \vee_R$$

$$\frac{\Gamma \vdash B \quad \Gamma, C \vdash A}{\Gamma, B \supset C \vdash A} \supset_L \qquad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \supset C} \supset_R$$

$$\frac{\Gamma, P[t/x] \vdash A}{\Gamma, \forall x.P \vdash A} \forall_L \qquad \frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} \forall_R *$$

$$\frac{\Gamma, P \vdash A}{\Gamma, \exists x.P \vdash A} \exists_L * \qquad \frac{\Gamma \vdash P[t/x]}{\Gamma \vdash \exists x.P} \exists_R$$

$$\frac{\Gamma' \vdash A'}{\Gamma \vdash A} \lambda \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash B} \bot_R$$

Figure 1. Higher-order Sequent Rules

reader to [5], and will limit ourselves to recapitulating the rules of inference, in Fig. 1, λ being $\beta\eta$ and structural rules, as contraction and weakening, being implicitly assumed. Rules do not include the cut rule:

$$\frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A} \text{Cut}$$

When we mean a proof within the rules of Fig. 1, we use the turnstyle $\vdash^*$, and use $\vdash$ when we allow the cut rule. In the rest of the paper, we will consider a fixed language S for ICTT, i.e. for each type a set of constants.

3 From Semi-valuations to Valuations: The Takahashi-Schütte lemma

We borrow the name semi-valuation from Takahashi and Schütte [22, 23, 24] also used by Andrews [1] to describe a partial interpretation of formulae in type theory that satisfies certain consistency properties, although our adaptation to the case of intuitionistic type-theory and Heyting algebras requires a considerable reworking of the definitions. Our formulation starts from constraints giving both positive and negative partial information: semivaluations consist of a *pair* of approximations to a model, which specify lower and upper bounds to the desired full interpretation. This is an abstraction of the way both positive and negative information from a Hintikka set is used to build a model for type theory in [5].

In *op. cit.* partial valuations are defined on the carrier of type o of an arbitrary typed applicative structure, and are shown, in this general setting, to extend to a full valuation without appeal to induction on subformulae which is not possible in an impredicative theory. The admissibility of cut then follows as an easy corollary. The cited result includes partial valuations on term models as a special case. Since this is all we need here, we will restate the main definitions and results for open terms only. We will also restrict attention to global models, defined below, since they are sufficient for the partial valuations chosen later in the paper to give a constructive proof of cut-admissibility.

3.1 Applicative Structures and Global Models

We will make use of the notion of applicative structures, a well-known semantic framework for the simply-typed lambda calculus, first introduced systematically by H. Friedman in [9], although obviously implicit in one form or another in [11, 15, 20]. (See also [17] for a detailed discussion.)

DEFINITION 1. A typed applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}\rangle$ consists of an indexed family $\mathsf{D} = \{\mathsf{D}_\alpha\}$ of sets D_α for each type α, an indexed family App of functions $\mathsf{App}_{\alpha,\beta} : \mathsf{D}_{\beta\alpha} \times \mathsf{D}_\alpha \to \mathsf{D}_\beta$ for each pair (α, β) of types, and an (indexed) interpretation function $\mathsf{Const} = \{\mathsf{Const}_\alpha\}$ taking constants of each type α to elements of D_α.

An assignment φ is a function from the free variables of the language into D which respects types, and which allows us to give meaning to open terms. Given a typed applicative structure D, an *environmental model* consists of a total function $\{\!\!\{\ \}\!\!\}_\varphi$ from the open terms of the language into D for each assignment φ respecting types, for which the following equalities hold:

$$
\begin{aligned}
\{\!\!\{c\}\!\!\}_\varphi &= \mathsf{Const}(c) && \text{for constants } c\\
\{\!\!\{x\}\!\!\}_\varphi &= \varphi(x) && \text{for variables } x\\
\{\!\!\{(MN)\}\!\!\}_\varphi &= \mathsf{App}(\{\!\!\{M\}\!\!\}_\varphi, \{\!\!\{N\}\!\!\}_\varphi)\\
\mathsf{App}(\{\!\!\{\lambda x_\alpha . M_\beta\}\!\!\}_\varphi, d) &= \{\!\!\{M\}\!\!\}_{\varphi[x:=d]}
\end{aligned}
$$

In the presence of extensionality, if an environmental model exists for a given assignment, it is unique, as the reader can show by proving the relevant substitution theorem.

So far we have only supplied semantics for the underlying typed lambda-calculus. Now we must interpret the logic as well, by adjoining a Heyting algebra and some additional structure to handle the logical constants and predicates.

DEFINITION 2. A Heyting applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \omega, \Omega \rangle$ for ICTT is a typed applicative structure with an associated Heyting algebra Ω and function ω from D_o to Ω such that for each f in $\mathsf{D}_{o\alpha}$, Ω contains the parametrized meets and joins

$$\bigwedge\{\omega(\mathsf{App}(f,d)) : d \in \mathsf{D}_\alpha\} \text{ and } \bigvee\{\omega(\mathsf{App}(f,d)) : d \in \mathsf{D}_\alpha\},$$

and the following conditions are satisfied:

$$\begin{array}{rcl}
\omega(\mathsf{Const}(\top_o)) & = & \top_\Omega \\
\omega(\mathsf{Const}(\bot_o)) & = & \bot_\Omega \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\wedge_{ooo}), d_1), d_2)) & = & \omega(d_1) \wedge \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\vee_{ooo}), d_1), d_2)) & = & \omega(d_1) \vee \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{App}(\mathsf{Const}(\supset_{ooo}), d_1), d_2)) & = & \omega(d_1) \rightarrow \omega(d_2) \\
\omega(\mathsf{App}(\mathsf{Const}(\Sigma_{o(o\alpha)}), f)) & = & \bigvee\{\omega(\mathsf{App}(f,d)) : d \in \mathsf{D}_\alpha\} \\
\omega(\mathsf{App}(\mathsf{Const}(\Pi_{o(o\alpha)}), f)) & = & \bigwedge\{\omega(\mathsf{App}(f,d)) : d \in \mathsf{D}_\alpha\}
\end{array}$$

By supplying an object Ω of truth values we are able to distinguish between denotations of formulae (elements d of D_o) and their truth-values $\omega(d) \in \Omega$. A definition, with suitable further restrictions on D_o, that identified D_o with Ω (*i.e.*, restricting ω to the identity function) might seem more natural but would make, for example, $A \wedge B$ indiscernible from $B \wedge A$ in the structure and thereby identify the truth values of $P_{oo}(A_o \wedge B_o)$ and $P_{oo}(B_o \wedge A_o)$. This identity holds neither in ICTT as presented here nor in the HOHH sub-system used in the λProlog programming language.

DEFINITION 3. A global model for ICTT is a total assignment-indexed function $\mathfrak{D} = \{\mathfrak{D}(\ \ )_\varphi : \varphi \text{ an assignment}\}$ into a Heyting applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \omega, \Omega \rangle$ which takes (possibly open) terms of type α into D_α and satisfies the environmental model conditions cited above, following Def.

1, as well as $\beta\eta$-conversion, that is to say:

$$\begin{aligned}
\mathfrak{D}(c)_\varphi &= \mathsf{Const}(c) && \text{for constants } c\\
\mathfrak{D}(x)_\varphi &= \varphi(x) && \text{for variables } x\\
\mathfrak{D}((MN))_\varphi &= \mathsf{App}(\mathfrak{D}(M)_\varphi, \mathfrak{D}(N)_\varphi) &&\\
\mathsf{App}(\mathfrak{D}(\lambda x_\alpha . M_\beta)_\varphi, d) &= \mathfrak{D}(M)_{\varphi[x:=d]} &&\\
\mathfrak{D}(M)_\varphi &= \mathfrak{D}(N)_\varphi && M \ \beta\eta\text{-equivalent to } N
\end{aligned}$$

The reader will note that we have added the requirement of $\beta\eta$ soundness to our definition of environmental models above. In the presence of functional extensionality (which we do not require) this is unnecessary, as our environmental models satisfy the substitution lemma, and are easily shown sound for beta and eta equivalence and uniquely determined by the assignment φ and the function Const.

Given a model $\mathfrak{D}$ and an assignment φ, we say that φ satisfies B in $\mathfrak{D}$ if $\omega(\mathfrak{D}(B_o)_\varphi) = \top_\Omega$; we abbreviate this assertion to $\mathfrak{D} \models_\varphi B_o$. We say B_o is valid in $\mathfrak{D}$ (equivalently, $\mathfrak{D} \models B_o$) if $\mathfrak{D} \models_\varphi B_o$ for every assignment φ. We abbreviate the truth-value $\omega(\mathfrak{D}(B_o)_\varphi)$ to $(B_o)^*_\varphi$. We also omit the subscript φ when our intentions are clear. We often use the word *model* just to refer to the mapping $(\_)^*$ from logical formulae to truth values in Ω.

Soundness of ICTT for Global Models

In the following we extend interpretations to sequents in a natural way.

DEFINITION 4. We define the meaning of a sequent in a model to be the truth-value in Ω given by:

$$(\Gamma \vdash \Delta)^* := (\bigwedge \Gamma \supset \Delta)^*$$

where $\bigwedge \Gamma$ signifies the conjunction of the elements of Γ and where we recall that, in an intuitionistic calculus, the consequent Δ is restricted to a single formula.

Note that $(\bigwedge \Gamma \supset \Delta)^* = \top$ if and only if $\top \leq (\bigwedge \Gamma \supset \Delta)^*$, which is to say $\top \leq (\bigwedge \Gamma)^* \to (\Delta)^*$, which by the condition on $\to$ is equivalent to $\top \wedge (\bigwedge \Gamma)^* \leq (\Delta)^*$, which in turn is equivalent to $(\bigwedge \Gamma)^* \leq (\Delta)^*$. We will abbreviate $(\bigwedge \Gamma)^*$ to $(\Gamma)^*$ and express the validity of the indicated sequent by $(\Gamma)^* \leq (\Delta)^*$ or, when referring to the environment, by $(\Gamma)^*_\varphi \leq (\Delta)^*_\varphi$ henceforth.

THEOREM 5 (Soundness). *If* $\Gamma \vdash A$ *is provable in* ICTT *then* $(\Gamma)^* \leq (A)^*$ *in every global model* $\mathfrak{E}$ *of* ICTT.

A proof can be found in [5]. Remind that $\vdash$ allows the cut rule.

A straightforward proof of completeness of ICTT for global models can be given *under the assumption that cut is admissible for ICTT* along the lines of [25, 5], i.e. by choosing Ω to be the Lindenbaum algebra of equivalence classes of formulae and then interpreting each formula as its own equivalence class. Just to show Ω is partially ordered, we need cut.

Since we are not assuming cut holds in ICTT we must proceed differently. We will choose the complete Heyting algebra Ω_{cfk} generated by "cut-free contexts", that is to say, contexts from which formulae can be proved without using cut. A partial valuation will be defined for this cHa, yielding an interpretation that establishes completeness and the admissibility of cut.

3.2 Semantic preliminaries

DEFINITION 6. Let Ω be a Heyting algebra. A global Ω **semivaluation** $\mathcal{V} = \langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \pi, \nu, \Omega \rangle$ consists of a typed applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const} \rangle$ together with a pair of maps $\pi : \mathsf{D}_o \longrightarrow \Omega$ and $\nu : \mathsf{D}_o \longrightarrow \Omega$, called the lower and upper constraints of $\mathcal{V}$, or the positive and negative constraints, satisfying the following:

1. For any $d \in \mathsf{D}_o$

$$\pi(d) \leq \nu(d)$$

2.

$$\begin{array}{rcl}
\pi(\top_o) & = & \top_\Omega \\
\pi(\bot_o) & = & \bot_\Omega \\
\pi(\mathsf{Const}(\wedge) \cdot A \cdot B) & \leq & \pi(A) \wedge_\Omega \pi(B) \\
\pi(\mathsf{Const}(\vee) \cdot A \cdot B) & \leq & \pi(A) \vee_\Omega \pi(B) \\
\pi(\mathsf{Const}(\supset) \cdot A \cdot B) & \leq & \pi(A) \rightarrow_\Omega \pi(B) \\
\pi(\mathsf{Const}(\Sigma_{o(o\alpha)}) \cdot f) & \leq & \bigvee\{\pi(f \cdot d) : d \in \mathsf{D}_\alpha\} \\
\pi(\mathsf{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) & \leq & \bigwedge\{\pi(f \cdot d) : d \in \mathsf{D}_\alpha\}
\end{array}$$

and

$$
\begin{array}{rcl}
\nu(\top_o) & = & \top_\Omega \\
\nu(\bot_o) & = & \bot_\Omega \\
\nu(\mathsf{Const}(\wedge) \cdot A \cdot B) & \geq & \nu(A) \wedge_\Omega \nu(B) \\
\nu(\mathsf{Const}(\vee) \cdot A \cdot B) & \geq & \nu(A) \vee_\Omega \nu(B) \\
\nu(\mathsf{Const}(\supset) \cdot A \cdot B) & \geq & \nu(A) \rightarrow_\Omega \nu(B) \\
\nu(\mathsf{Const}(\Sigma_{o(o\alpha)}) \cdot f) & \geq & \bigvee\{\nu(f \cdot d) : d \in \mathsf{D}_\alpha\} \\
\nu(\mathsf{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) & \geq & \bigwedge\{\nu(f \cdot d) : d \in \mathsf{D}_\alpha\}
\end{array}
$$

3. and the *consistency* or *separation* conditions

$$
\pi(\mathsf{Const}(\supset) \cdot B \cdot C) \wedge \nu(B) \leq \pi(C) \tag{1}
$$

$$
\pi(B) \rightarrow_\Omega \nu(C) \leq \nu(\mathsf{Const}(\supset) \cdot B \cdot C). \tag{2}
$$

REMARK 7. In this definition, the application operator App is denoted by the infix operator $\cdot$ for readability. The reader should note that some of these requirements are superfluous, i.e. follow from the others. The separation conditions and the first condition imply the $\supset$ requirements for both π and ν, as well as $\top$ requirement for π (resp. *bot* for ν) implies their counterpart for ν (resp. π).

The separation conditions abstract the properties of the weak and strong support sets[1] $\mathcal{H}_A^{\mathsf{T}}$ and $\mathcal{H}_A^{\neg\mathsf{F}}$ associated with a Hintikka set $\mathcal{H}$ in [5].

The definition of environment, and global structure remain the same for semivaluations. As with Heyting applicative structures, in the presence of an environment φ, a semivaluation $\mathcal{V}$ induces an interpretation $\mathfrak{V}_\varphi$ from open terms A to the carriers D as follows:

$\mathfrak{V}(c)_\varphi$	$= \mathsf{Const}(c)$	for constants c
$\mathfrak{V}(x)_\varphi$	$= \varphi(x)$	for variables x
$\mathfrak{V}(M)_\varphi$	$= \mathfrak{V}(N)_\varphi$	M eta-equivalent to N
$\mathfrak{V}((MN))_\varphi$	$= \mathsf{App}(\mathfrak{V}(M)_\varphi, \mathfrak{V}(N)_\varphi)$	
$\mathsf{App}(\mathfrak{V}(\lambda x_\alpha . M_\beta)_\varphi, d)$	$= \mathfrak{V}(M)_{\varphi[x:=d]}$	

This assignment induces a *pair* of partial, or semi-truth-value assignments $[\![\_]\!]_\varphi^\pi$ and $[\![\_]\!]_\varphi^\nu$ to terms A_o of type o given by

$$
\begin{array}{rcl}
\mathcal{V}[\![A]\!]_\varphi^\pi & = & \pi(\mathfrak{V}(A)_\varphi) \\
\mathcal{V}[\![A]\!]_\varphi^\nu & = & \nu(\mathfrak{V}(A)_\varphi)
\end{array}
$$

[1]whose formulation is due to Chad Brown.

THEOREM 8. *Given an* Ω, S*-semivaluation* $\mathcal{V} = \langle \mathsf{D}, \cdot, \mathsf{Const}, \pi, \nu, \Omega \rangle$*, there is a model* $\mathfrak{D} = \langle \hat{\mathsf{D}}, \odot, \hat{\mathsf{C}}, \omega, \Omega \rangle$ *extending* $\mathcal{V}$ *in the following sense: for all closed terms* A_o

$$\mathcal{V}[\![A]\!]^{\pi} \leq \omega(\mathfrak{D}(A)) \leq \mathcal{V}[\![A]\!]^{\nu}.$$

Furthermore, there is a surjective indexed map $\delta : \hat{\mathsf{D}} \longrightarrow \mathsf{D}$ *such that for any* $\hat{d} \in \hat{\mathsf{D}}_o$

$$\pi(\delta(\hat{d})) \leq \omega(\hat{d}) \leq \nu(\delta(\hat{d}))$$

Proof. We recall from the constructions in [24, 1, 5] that a $\mathcal{V}$-complex of a given type γ is an ordered pair $\langle A_\gamma, u \rangle$ where A_γ is a term (of type γ) in normal form, and u is a truth value, in our case a member of a cHa, which we can think of as a candidate truth value for the desired valuation (i.e. model) $\mathfrak{D}$. As in the cited works, the carriers $\hat{\mathsf{D}}$ of this model $\langle \hat{\mathsf{D}}, \hat{\mathsf{C}}, \odot, \omega, \Omega \rangle$ will be sets of such $\mathcal{V}$ -complexes, defined by induction on the *type* structure, as follows:

- $\hat{\mathsf{D}}_o = \{\langle d, u \rangle : d \in \mathsf{D}_o \text{ and } \pi(d) \leq u \leq \nu(d)\}$
- $\hat{\mathsf{D}}_\iota = \{\langle m, \iota \rangle : m \in \mathsf{D}_\iota\}$
- $\hat{\mathsf{D}}_{\beta\alpha} = \{\langle m, \mu \rangle : m \in \mathsf{D}_{\beta\alpha}, \mu : \mathsf{D}_\alpha \longrightarrow \mathsf{D}_\beta,$ and for each $\langle A, a \rangle \in \mathsf{D}_\alpha, \mu\langle A, a \rangle = \langle m \cdot A, r \rangle$ for some $r\}$.
- Application is given by $\langle M, m \rangle \odot \langle A, a \rangle = m\langle A, a \rangle$.
- Define $\omega : \hat{\mathsf{D}}_o \longrightarrow \Omega$ by projection on the second coordinate.

Projections on the first and second coordinates are noted d^1 and d^2, respectively. As in [5], we can define a *selector* function $\rho : \mathsf{D} \longrightarrow \hat{\mathsf{D}}^2$ by induction on types, to show that for every type α and every $M \in \mathsf{D}_\alpha$ there is a $\rho(M)$ such that $\langle M, \rho(M) \rangle \in \hat{\mathsf{D}}_\alpha$.

Notice that in the D_o base case, we have a degree of freedom: we can choose $\rho(M) = \pi(M)$ as well $\rho(M) = \nu(M)$. This choice can be uniform (and arbitrary) or depend on M, as we shall see later.

Now we show how to define the assignment of denotations to logical and non-logical constants.

$$\begin{array}{rl}
\hat{\mathsf{C}}(\top_o) = & \langle \mathsf{Const}(\top_o), \top_\Omega \rangle \\
\hat{\mathsf{C}}(\bot_o) = & \langle \mathsf{Const}(\bot_o), \bot_\Omega \rangle \\
\hat{\mathsf{C}}(c_\alpha) = & \langle \mathsf{Const}(c_\alpha), \rho(\mathsf{Const}(c_\alpha)) \rangle \text{ for non-logical constants } c_\alpha. \\
\hat{\mathsf{C}}(\wedge) = & \langle \mathsf{Const}(\wedge), \boldsymbol{\lambda}\langle B, b \rangle . \langle \mathsf{Const}(\wedge) \cdot B, \boldsymbol{\lambda}\langle D, d \rangle . \langle \mathsf{Const}(\wedge) \cdot B \cdot D, b \wedge_\Omega d \rangle \rangle \rangle \\
\hat{\mathsf{C}}(\supset) = & \langle \mathsf{Const}(\supset), \boldsymbol{\lambda}\langle B, b \rangle . \langle \mathsf{Const}(\supset) \cdot B, \boldsymbol{\lambda}\langle D, d \rangle . \langle \mathsf{Const}(\supset) \cdot B \cdot D, b \rightarrow_\Omega d \rangle \rangle \rangle \\
\hat{\mathsf{C}}(\Sigma) = & \langle \mathsf{Const}(\Sigma), \boldsymbol{\lambda}\langle M, m \rangle . \langle \mathsf{Const}(\Sigma) \cdot M, \bigvee_{\hat{d} \in \hat{\mathsf{D}}_\alpha} (m\hat{d})^2 \rangle \rangle .
\end{array}$$

where Σ abbreviates $\Sigma_{o(o\alpha)}$. The $\vee$ and Π cases are similar, and left to the reader.

We now need to show that $\hat{\mathsf{C}}$ is well-defined. This is where the separation conditions play a key role. We will work a few cases.

$\hat{\mathsf{C}}(\wedge)$ What we must show here is that if $\langle B, b\rangle$ and $\langle D, d\rangle$ are in $\hat{\mathsf{D}}_o$ then so is $\langle \mathsf{Const}(\wedge)\cdot B\cdot D, b\wedge_\Omega d\rangle$. That is to say, if we are given that $\pi(B)\leq b\leq \nu(B)$ and $\pi(D)\leq d\leq\nu(D)$ then $b\wedge_\Omega d$ lies between $\pi(\mathsf{Const}(\wedge)\cdot B\cdot D)$ and $\nu(\mathsf{Const}(\wedge)\cdot B\cdot D)$. Since $\wedge$ is monotone in both arguments this follows immediately from the defining properties of upper and lower constraints.

The argument for $\vee$ is similar.

$\hat{\mathsf{C}}(\supset)$ We must show that the second component $\hat{\mathsf{C}}(\supset)^2$, namely the term $\boldsymbol{\lambda}\langle B, b\rangle.\langle\mathsf{Const}(\supset)\cdot B, \boldsymbol{\lambda}\langle D, d\rangle.\langle\mathsf{Const}(\supset)\cdot B\cdot D, b\rightarrow_\Omega d\rangle\rangle$ maps a pair of members of $\hat{\mathsf{D}}_o$ to $\hat{\mathsf{D}}_o$. If we are given two members $\langle B, b\rangle$ and $\langle D, d\rangle$ of $\hat{\mathsf{D}}_o$, then we know $\pi(B)\leq b\leq\nu(B)$ and similarly $\pi(D)\leq d\leq\nu(D)$. But then, abbreviating $\mathsf{Const}(\supset)\cdot B\cdot D$ to $B\supset D$, we have $\pi(B\supset D)\wedge b\leq \pi(B\supset D)\wedge\nu(B)$. By the first separation axiom, $\pi(B\supset D)\wedge b\leq\pi(D)\leq d$. But then $\pi(B\supset D)\leq b\rightarrow d$.

Furthermore $b\rightarrow d\leq\pi(B)\rightarrow\nu(D)$ since Heyting implication is antitone (contravariant) in its first argument and monotone in its second. By the second separation axiom (2) $b\rightarrow d\leq\nu(B\supset D)$,as we wanted to show.

The Π and Σ cases are both monotone in the relevant arguments, and are easy. The surjective map δ in the conclusion of the theorem is just projection of $\mathcal{V}$-complexes onto their first component.

The rest of the proof that $\mathfrak{D}$ is a model follows just like the proof for the model constructed in [5]. ■

4 Completeness and cut elimination - the ICTT case

From Thm. 8, deriving a (cut-free) completeness theorem for ICTT requires a complete Heyting algebra Ω and an Ω, S semivaluation. We first give the definition of Ω_{cfk}, the Heyting algebra of cut-free contexts, which is very different from the one given in [5].

4.1 The cut-free contexts Heyting algebra

We first define what is a cut-free context, in the same way as Okada [19, 18].

DEFINITION 9 (outer value). Let A be a closed formula. We let the *outer value* of A be:

$$[\![A]\!] = \{\Gamma \mid \Gamma \vdash^* A\}$$

So, an outer value $[\![A]\!]$ is the set of contexts proving A without cut (cut-free contexts). With this, we build Ω_{cfk}.

DEFINITION 10 (Ω_{cfk}). We let $|\Omega|$ to be the least set of sets of (finite) contexts generated by $[\![A]\!]$ for any formula A, and closed under arbitrary intersection. It is ordered by inclusion. Then define meets and joins on $|\Omega|$ as follows

- $\bigwedge$ = arbitrary intersection, just set-theoretic intersections.
- $\bigvee$ = arbitrary pseudo-union, that is to say

  $$\bigvee S = \bigcap \{c \in |\Omega| : c \geq S\}$$

  where $c \geq S$ means $\forall s \in S\, c \geq s$

REMARK 11. If we expand this definition a little bit, we have:

- $\top_\Omega$ is the set of all finite contexts. It is as well $[\![\top_o]\!]$ since any context proves $\top_o$ without using cut.
- $\bot_\Omega$ is the intersection of all outer values. Equivalently, it is $[\![\bot_o]\!]$ since if $\Gamma \vdash^* \bot_o$ we can prove $\Gamma \vdash^* A$ for any A. In particular, $\bot_\Omega$ is not empty.

Notice that an element $c \in \Omega$ (by which we mean $c \in |\Omega|$) can always be written as an element of the form $\bigcap [\![A_i \mid i \in \Lambda]\!]$. So we can simplify a little bit the definition of union:

LEMMA 12 (Simplification of the definition). *We may express suprema directly in terms of generating sets:*

- $\bigvee\{a_i, i \in I\} = \bigcap\{[\![A]\!] \mid \bigcup\{a_i, i \in I\} \subseteq [\![A]\!]\}$
- $a \vee_\Omega b = \bigcap\{[\![A]\!] \mid a \cup b \subseteq [\![A]\!]\}$

Proof. Each one of the c mentioned before is of the form $\bigcap\{c_i, i \in J\}$. ■

Taking $a \to b = \bigvee\{x : x \wedge a \leq b\}$, the resulting structure $\Omega = \langle |\Omega|, \bigvee, \bigwedge, \to\rangle$ (also written Ω_{cfk}, when ambiguity may arise) is a complete Heyting algebra. One must show that the $\wedge \bigvee$ distributivity law holds [25].

First we show that for each member $a = \bigcap_i [\![A_i]\!]$ of Ω

$$a \cap \bigvee S \leq \bigvee a \cap S$$

where $a \cap S$ means $\{a \cap s : s \in S\}$. Unfolding the definitions and using Lem. 12 above, the desired conclusion is equivalent to

$$a \cap \bigcap\{[\![B]\!] : [\![B]\!] \geq S\} \subseteq \bigcap\{[\![D]\!] : [\![D]\!] \geq a \cap S\} \qquad (3)$$

where $x \geq S$ abbreviates $\forall s \in S (s \subseteq x)$. Suppose the context Γ is a member of the left hand side, i.e. for each i we have $\Gamma \vdash^* A_i$ and $\Gamma \vdash^* B$ for every B such that $[\![B]\!] \geq S$.

Let D be a formula such that $[\![D]\!] \geq a \cap S$. We must show $\Gamma \vdash^* D$ to conclude.

Let Δ ba a context such that $\Delta \in s$ for some $s \in S$. By weakening $\Delta, \Gamma \vdash^* A_i$ for each i, i.e. $\Delta, \Gamma \in a$ and by the same reasoning $\Delta, \Gamma \in s$. By definition of D, we have $\Delta, \Gamma \vdash^* D$. Hence $\Delta \vdash^* \bigwedge \Gamma \supset D$. Since this is valid for any s, we have shown $[\![\Gamma \supset D]\!] \geq S$.

But then, $\Gamma \vdash^* \Gamma \supset D$ by assumption on Γ. By Kleene's Lem. 33 below and contraction on the formulae in Γ we have $\Gamma \vdash^* D$, which shows Γ is a member of the right-hand-side of 3, which proves the claim.

The other direction follows, by elementary lattice theory: for any $s \in S$ it is the case that $a \cap \bigvee S \geq a \cap s$. Now take the supremum of $a \cap s$ over all $s \in S$.

4.2 A semivaluation π and ν

Now, we need to give a definition of an Ω semivaluation to have the right to apply Thm. 8. For this, we need the following definition:

DEFINITION 13 (closure). Let S be a set of contexts, we define its closure by:

$$cl(S) = \bigcap\{[\![A]\!] \mid S \subseteq [\![A]\!]\}$$

It is the least element of Ω containing S. We also write, for a single context Γ, $cl(\Gamma)$ to mean $cl(\{\Gamma\})$.

REMARK 14. Notice that $cl(A) \subseteq d$ is equivalent to $A \in d$. Indeed, $A \in cl(A)$ and $cl(A)$ is the l.u.b. of A. The closure operator can also be understood as the *set of contexts admitting cut with all the elements of S* as shown in the following lemma.

LEMMA 15. *Let A be a formula. Then the four following formulations are equivalent:*

(i) $cl(A) = \bigcap\{[\![B]\!] \mid A \in [\![B]\!]\}$

(ii) $cl(A) = \{\Gamma \mid \Gamma \vdash^* B$ *whenever* $A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff* $\Gamma \vdash^* A$ *and given any proof* $A \vdash^* B$*, a proof of* $\Gamma \vdash^* B$ *is derivable.*

(iii) $cl(A) = \{\Gamma \mid \Gamma \vdash^* B$ *whenever* $\Gamma, A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff* $\Gamma \vdash^* A$ *and given any proof* $\Gamma, A \vdash^* B$ *a proof of* $\Gamma \vdash^* B$ *is derivable.*

(iv) $cl(A) = \{\Gamma \mid \Delta, \Gamma \vdash^* B$ *whenever* $\Delta, A \vdash^* B\}$. *Equivalently,* $\Gamma \in cl(A)$ *iff* $\Gamma \vdash^* A$ *and given any proof* $\Delta, A \vdash^* B$ *a proof of* $\Delta, \Gamma \vdash^* B$ *is derivable.*

Cases (ii) – (iv) *can be summarized as follows:* Γ *admits cuts with* A*, hence the terminology "*Γ *is* A*-cuttable".*

Proof. Denoting $cl(A)$ as defined at point (x) as (x) itself, we have:

- $(i) = (ii)$ is just an unfolding of $[\![B]\!]$. Moreover $A \in [\![A]\!]$, thus $\Gamma \vdash^* A$ has to hold.
- $(ii) \subseteq (iii)$. Let $\Gamma \in (ii)$, and B such that $\Gamma, A \vdash^* B$. Show that $\Gamma \vdash^* B$. By $\wedge_L$ rules and a $\supset_R$ rule, we have a proof of the sequent $A \vdash^* (\bigwedge \Gamma) \supset B$. Therefore, by hypothesis, $\Gamma \vdash^* (\bigwedge \Gamma) \supset B$. Using Kleene's inversion lemma Lem. 33 below, we get a proof of the sequent $\Gamma, \Gamma \vdash^* B$ that we contract.
- $(iii) \subseteq (iv)$. Let $\Gamma \in (iii)$ and B such that $\Delta, A \vdash^* B$. By weakening, $\wedge_L$ and $\supset_R$ rules, this yields a proof of the sequent $\Gamma, A \vdash^* \bigwedge \Delta \supset B$. $\Gamma \vdash^* \bigwedge \Delta \supset B$ then holds by hypothesis, and we conclude by applying Kleene's inversion lemma Lem. 33.
- $(iv) \subseteq (ii)$. Let $\Gamma \in (iv)$ and B such that $A \vdash^* B$. Taking $\emptyset$ for Δ in (iv) shows $\Gamma \vdash^* B$.

■

We will not pay much attention to the $\Gamma \vdash^* A$ statement. It is an immediate consequence of the A-cuttability statement, since $A \vdash^* A$ trivially. We shall use any of those formulations, depending on our need. Now we are ready to give the semivaluation we work with:

DEFINITION 16 (the semivaluation). Let the typed applicative structure $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const} \rangle$ be the open term model: we take carriers D_α to be open terms in normal form of the appropriate type, application $A \cdot B = [AB]$, and we interpret constants as themselves. For any formula A, we define:

$$\begin{array}{rcl} \pi(A) & = & cl(A) \\ \nu(A) & = & [\![A]\!] \end{array}$$

LEMMA 17. $\langle \mathsf{D}, \mathsf{App}, \mathsf{Const}, \pi, \nu, \Omega_{\mathsf{cfk}} \rangle$ *is a semivaluation in the sense of Def. 6.*

Proof. We have to check every statement of Def. 6, with respect to the open term model.

- $cl(A) \subseteq [\![A]\!]$. By Rem. 14, this amounts to showing $A \in [\![A]\!]$. This holds since $A \vdash^* A$.

- $cl(\top_o) = \top_\Omega$. The direct inclusion is immediate since $\top_\Omega$ is the greatest element. For the converse, we have to show that any context is $\top_o$-cuttable. So consider a proof of $\top_o \vdash^* A$ for some A. The only possible rules we can use on $\top_o$ besides contraction, weakening and conversion is the axiom. We can always replace it by:

$$\frac{}{\vdash \top_o}\ \top\text{-right}$$

  Hence, $\vdash^* A$, and, by weakening, $\Gamma \vdash^* A$ for any context Γ.

- $cl(\bot_o) = \bot_\Omega$, since $\bot_\Omega \subseteq cl(\bot) \subseteq [\![\bot]\!] = \bot_\Omega$: $\bot_\Omega$ is the least element, from a previous point and from Rem. 11.

- $cl(A \wedge B) \leq cl(A) \cap cl(B)$. This amounts to showing $A \wedge B \in cl(A) \cap cl(B)$. We prove that $A \wedge B$ is A-cuttable. Consider a proof of $A \vdash^* C$. We construct the following proof:

$$\dfrac{\dfrac{A \vdash^* C}{A, B \vdash^* C}\ \text{weak}}{A \wedge B \vdash^* C}\ \wedge_L$$

  Hence, $A \wedge B \in cl(A)$. On the same way, $A \wedge B \in cl(B)$ and the claim is proved.

- $cl(A \vee B) \subseteq cl(A) \vee_\Omega cl(B)$. It suffices to show $A \vee B \in cl(A) \vee_\Omega cl(B)$. Let C be such that $cl(A) \cup cl(B) \subseteq [\![C]\!]$. Then $A \in [\![C]\!]$ and $B \in [\![C]\!]$. Therefore, the proof

$$\frac{A \vdash^* C \qquad B \vdash^* C}{A \vee B \vdash^* C}\ \vee_L$$

  shows that $A \vee B \in [\![C]\!]$. This holds for any such C, hence $A \vee B \in cl(A) \vee_\Omega cl(B)$.

- $cl(A \supset B) \subseteq cl(A) \to cl(B)$ is a consequence of $cl(A \supset B) \wedge [\![A]\!] \subseteq cl(B)$ (proved later) as said in Rem. 7.

- $cl(\Sigma.f) \subseteq \bigvee\{cl(ft) \mid t \in \mathcal{T}_\alpha\}$ (where α is the suitable type). Equivalently, $\Sigma.f \in \bigvee\{cl((ft)) \mid t \in \mathcal{T}_\alpha\}$. Let t be a variable y of type α that is fresh for f. We prove that $\Sigma.f$ is fy-cuttable. Assume to have a proof $fy \vdash^* C$. The proof:

$$\dfrac{fy \vdash^* C}{\Sigma.f \vdash^* C}\ \exists_L$$

justifies the fy-cuttability. Hence $\Sigma.f \in cl(fy)$, and it is in the supremum.

- $\Pi.f \in \bigwedge\{cl(ft), t \in \mathcal{T}_\alpha\}$. Let t be a term of type α. The proof:

$$\dfrac{ft \vdash^* C}{\Pi.f \vdash^* C}\ \forall_L$$

shows that $\Pi.f$ is ft-cuttable for any t.

- $[\![\top_o]\!] = \top_\Omega$ and $[\![\bot_o]\!] = \bot_\Omega$ hold both by definition, from Rem. 11.

- $[\![A \wedge B]\!] \supseteq [\![A]\!] \wedge_\Omega [\![B]\!]$. Let Γ such that $\Gamma \vdash^* A$ and $\Gamma \vdash^* B$. The proof:

$$\dfrac{\Gamma \vdash^* A \qquad \Gamma \vdash^* B}{\Gamma \vdash^* A \wedge B}\ \wedge_R$$

shows the claim.

- $[\![A \vee B]\!] \supseteq [\![A]\!] \vee_\Omega [\![B]\!]$. We show $[\![A \vee B]\!] \supseteq [\![A]\!]$. Let $\Gamma \in [\![A]\!]$. The proof:

$$\dfrac{\Gamma \vdash^* A}{\Gamma \vdash^* A \vee B}\ \vee_R$$

shows that $\Gamma \in [\![A \vee B]\!]$. Hence $[\![A \vee B]\!]$ is an upper bound for $[\![A]\!]$ and $[\![B]\!]$, and the claim is proved.

- $[\![A \supset B]\!] \supseteq [\![A]\!] \to_\Omega [\![B]\!]$ is a consequence of $cl(A) \to [\![B]\!] \subseteq [\![A \supset B]\!]$ (proved later) as said in Rem. 7.

- $[\![\Sigma.f]\!] \supseteq \bigvee\{[\![ft]\!], t \in \mathcal{T}_\alpha\}$. Let t be any term of type α. Let $\Gamma \in [\![ft]\!]$. The proof:

$$\dfrac{\Gamma \vdash^* ft}{\Gamma \vdash^* \Sigma.f}\ \exists_R$$

shows that $[\![\Sigma.f]\!]$ is an upper bound for any $[\![ft]\!]$, hence for their supremum as well.

- $[\![\Pi.f]\!] \supseteq \bigwedge\{[\![ft]\!], t \in \mathcal{T}_\alpha\}$. Let $\Gamma \in \bigwedge\{[\![ft]\!], t \in \mathcal{T}_\alpha\}$. Let y be a fresh variable with respect to Γ and f. In particular, $\Gamma \in [\![fy]\!]$. The proof

$$\frac{\Gamma \vdash^* fy}{\Gamma \vdash^* \Pi.f}\ \forall_R$$

shows that $\Gamma \in [\![\Pi.f]\!]$.

- $cl(B \supset C) \wedge_\Omega [\![B]\!] \subseteq cl(C)$. Let $\Gamma \in cl(B \supset C) \cap [\![B]\!]$. We must show the C-cuttability of Γ. Consider a proof of $C \vdash^* D$. Since $\Gamma \vdash^* B$:

$$\frac{\Gamma \vdash^* B \qquad \Gamma, C \vdash^* D}{\Gamma, B \supset C \vdash^* D}\ \supset_L$$

By $B \supset C$-cuttability of Γ we get $\Gamma \vdash^* D$.

- $cl(B) \rightarrow_\Omega [\![C]\!] \subseteq [\![B \supset C]\!]$. Let $\Gamma \in cl(B) \rightarrow [\![C]\!]$ and show $\Gamma \vdash^* B \supset C$. Indeed, since $\Gamma \in cl(B) \rightarrow [\![C]\!]$, we have: $cl(\Gamma) \cap cl(B) \subseteq [\![C]\!]$. Furthermore from Rem. 14, $\Gamma \in cl(\Gamma)$ and $B \in cl(B)$. It follows by weakenings that Γ, B belongs to both. Hence $\Gamma, B \in [\![C]\!]$, and we can derive the desired proof:

$$\frac{\Gamma, B \vdash^* C}{\Gamma \vdash^* B \supset C}\ \supset_R$$

■

4.3 Completeness and cut elimination of ICTT

We now have all the results needed to establish completeness.

THEOREM 18 (cut-free completeness of ICTT). *Let Γ be a context and A be a formula. Assume that for any global model we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. We apply Thm. 8 with Heyting algebra Ω_{cfk} given in Def. 10 and the semivaluation π, ν of Def. 16. We get, from Rem. 14, by Thm. 8 and by hypothesis that:

$$\Gamma \in cl(\Gamma) \subseteq \Gamma^* \subseteq A^* \subseteq [\![A]\!]$$

Hence, the sequent $\Gamma \vdash A$ has a cut-free proof. ■

As an immediate corollary, we have:

COROLLARY 19 (constructive cut elimination for ICTT). *Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in* ICTT*, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively. ■

5 Adding non-logical axioms

Now, we allow a more liberal notion of proof, with *non-logical axioms*.

DEFINITION 20. A non-logical axiom is a sequent $A \vdash B$. A proof with non-logical axioms is a proof whose leaves are either a proper axiom rule, or a non-logical axiom and allowing the use of axiomatic cuts.

Assuming that $A \vdash B$ is a non-logical axiom, an axiomatic cut is the following implicit cut rule

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

In the sequel, we will work with a given set (potentially infinite) of non-logical axioms, and the proof system will be a proof ICTT with non-logical axioms. This syntactical proof system will be called L_{nla}.

The non-logical axiom and the axiomatic cut rules overlap a little bit:

LEMMA 21. *In* ICTT *with non-logical axioms, when the cut rule is allowed, one can simulate axiomatic cuts. Conversely, with axiomatic cuts one can simulate the non-logical axioms, with or without the cut rule.*

Proof. For the first statement, replace any axiomatic cut by:

$$\dfrac{\text{cut}\ \dfrac{\Gamma \vdash A \qquad \dfrac{}{A \vdash B}\ \text{non-logical axiom}}{\Gamma \vdash B} \qquad \Gamma, B \vdash C}{\Gamma \vdash C}\ \text{cut}$$

For the second statement, replace a non-logical axiom $\Gamma, A \vdash B$ by:

$$\dfrac{\dfrac{}{\Gamma, A \vdash A} \qquad \dfrac{}{\Gamma, B \vdash B}}{\Gamma, A \vdash B}\ \text{axiomatic cut}$$

■

We show in this section that we still have, by the same means, cut elimination in ICTT with non-logical axioms, but that we can not, in the general setting, eliminate axiomatic cuts. First, we need another, unsurprising, notion of model:

DEFINITION 22 (models for non-logical axioms). A global model for ICTT (Def. 3) is a model of the non-logical axioms if and only if $(A)^* \leq (B)^*$ for any non-logical axiom $A \vdash B$.

In the sequel, we will only be interested in such models.

THEOREM 23 (Soundness of ICTT with non-logical axioms). *If* $\Gamma \vdash A$ *in* ICTT *with non-logical axioms, then* $\Gamma^* \leq A^*$ *in any global model of the non-logical axioms.*

Proof. We assume not to have axiomatic cuts by Lem. 21: we replace them by usual cuts. The proof is by the very same induction as the one of Thm. 5. The only additional case is the case of a non-logical axiom $A \vdash B$, trivial, since we assumed the model to be a model of the non-logical axioms. ∎

Now we work towards a proof of a cut-free completeness theorem for ICTT with non-logical axioms. Cut-free means free of cuts, but not of axiomatic cuts, that we will not be able to remove.

5.1 Completeness and cut elimination in presence of axioms

As well as we defined, in ICTT, the complete Heyting algebra of cut-free contexts Ω_{cfk} (Def. 10), we can define Ω_{cfk} with respect to provability in L_{nla}, i.e. ICTT with non-logical axioms. Def. 10 does *not* depend on the syntactic system we work with. Of course the contexts in $[\![C]\!]$ depend on the logic we are in: ICTT or ICTT with non-logical axioms (L_{nla}). So both algebras are different, but generated exactly the same way. To distinguish between both algebras, we will speak about $\Omega(\mathrm{L})$ and $\Omega(\mathrm{L}_{nla})$.

We can build a semivaluation, with respect to provability in L_{nla}, exactly in the same way as in Def. 16: we can check that the proof of Lem. 17 does not depend on the presence or absence of non-logical axioms and axiomatic cuts. As well, the A-cuttability notion (Lem. 15) used there does not depend on ICTT. It appeals to Kleene's Lem. 33 below, that remains valid.

Since the Takahashi-Schütte lemma (Thm. 8) does not at all depend on the syntactic system (it requires only a semivaluation), we can build an interpretation, generating V-complexes [1, 24, 21] as well.

So we get a model $\mathfrak{D}_{nla}$ over the cHa $\Omega(\mathrm{L}_{nla})$ exactly in the same way as in Sec. 4. The only thing to check is:

LEMMA 24. *The global model* $\mathfrak{D}_{nla}$ *is a model of the non-logical axioms.*

Proof. Let $A \vdash B$ be a non-logical axiom. Let's show that $A^* \subseteq B^*$. We know that $A^* \subseteq [\![A]\!]$ and that $cl(B) \subseteq B^*$ from the Takahashi-Schütte Thm. 8. So it is sufficient to show $[\![A]\!] \subseteq cl(B)$. Let Γ such that $\Gamma \vdash^* A$. We show that Γ is B-cuttable. So, assume given a formula C and a proof of $\Gamma, B \vdash^* C$. We can build the following proof of $\Gamma \vdash^* C$:

$$\frac{\Gamma \vdash^* A \qquad \Gamma, B \vdash^* C}{\Gamma \vdash^* C} \text{ axiomatic cut}$$

which yields the desired conclusion. ■

Therefore, we have the following proof of completeness:

THEOREM 25 (cut-free completeness of ICTT with nla). *Consider any set of non-logical axioms. Let Γ be a context and A be a formula. Assume that for any global model of the non-logical axioms, we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. As Thm. 18. $\Omega(\mathrm{L}_{nla})$ is a global model of the non-logical axioms. ■

As well, we have the cut elimination theorem as a corollary:

COROLLARY 26 (constructive cut elimination for ICTT with non-logical axioms). *Consider any set of non-logical axioms. Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in* ICTT*, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively. ■

6 ICTT and rewriting cuts

Here, we assume we have non-logical axioms as well (Def. 20), as in the previous section but with the additional assumption that the non-logical axioms satisfy a sign-preserving condition (given below). We obtain a constructive semantic cut elimination theorem that guarantees the existence of a cut-free proof with the following *rewriting cuts* only, which are a restriction of axiomatic cuts.

Before proving the strengthened cut elimination theorem, we state some useful results, and we begin by recalling briefly terminology of rewriting.

6.1 Rewrite rules

DEFINITION 27. Let $\mathcal{R} = \{l_i \rightarrow r_i : i \in I\}$ be a rewrite system where all the left and right members have type o.

A proposition C is said to $\mathcal{R}$-rewrite to C' if for some i and some $D \equiv_\lambda C$ such that there is a redex in D matching an instance of l_i, via a unifier θ, D' is D where this redex is replaced by θr_i and $D' \equiv_\lambda C'$.

We use the notation $\rightarrow^*$ to denote the transitive, reflexive closure of the $\mathcal{R}$ relation $\rightarrow$ and $\equiv_\mathcal{R}$ the congruence it generates.

A rewrite system $\mathcal{R}$ is confluent if and only if for any two formulae $A \equiv_\mathcal{R} B$, there is a formula C such that $A \rightarrow^* C$ and $B \rightarrow^* C$. A rewrite system $\mathcal{R}$ is an atomic system if every antecedent A with $A \rightarrow B \in \mathcal{R}$ is an atomic formula of type o.

The reader should consult e.g. [2] for more details on term rewriting. Notice that $\equiv_\mathcal{R}$ contains $\equiv_\lambda$ and that here we are interested only in propositional rewriting, which is where interaction between rewriting and sequent rules can be delicate (see Deduction Modulo [12]).

LEMMA 28 (Main connective). *Let $\{A_i \rightarrow B_i\}$ be an atomic and confluent rewrite system. Let C, D be non atomic formulae. If $C \equiv_\mathcal{R} D$, they have the same main connective, and their immediate subformulae are congruent.*

Proof. By confluence, we can find a E such that $C \rightarrow^* E$ and $D \rightarrow^* E$, and rewriting occur only on atomic formulae by the atomicity condition. Formally, this is done by induction on the length of rewriting paths. For more details, see [12]. ■

6.2 From rewrite rules to ICTT

There are many ways to add rewrite rules to ICTT, for instance, one can try to define a Deduction Modulo [7, 6] within the ICTT frame. Instead of that, we constrain the axiomatic cuts of L_{nla} to have a rewriting form.

6.3 Rewriting cuts

DEFINITION 29 (From rewrite rules to an axiomatic system). Let $\mathcal{R}$ be a propositional rewrite system consisting of rules of the form $A \rightarrow B$ with A, B terms of type o. We consider the associated set of non logical axioms (i.e. new initial rules) to be all the sequents $\sigma A \vdash \sigma B$ and $\sigma B \vdash \sigma A$ where $A \rightarrow B \in \mathcal{R}$ and σ is some substitution: we add all the instantiations of rewrite rules, in both ways.

DEFINITION 30 (Rewriting cut). Let $A \vdash B$ a non-logical axiom. A rewriting cut is an axiomatic cut of one of the following forms:

$$\frac{\Gamma \vdash A \qquad \overline{\Gamma, A \vdash B}\ \mathsf{nlAx}}{\Gamma \vdash B} \qquad\qquad \frac{\mathsf{nlAx}\ \overline{\Gamma, A \vdash B} \qquad \Gamma, B \vdash C}{\Gamma, A \vdash C}$$

We define the logic L^{nla} to be L_{nla} with axiomatic cuts restricted in this way.

This restriction makes rewriting cuts almost inoffensive, since the formula we cut on has to be immediately proved in one premise. That is very close to a rewriting of A to B (on the right hand side) and of B to A (on the left hand side).

As in Lem. 21, one can simulate the non-logical axiom rule, even in a cut-free setting, with a rewrite cut. Of course the non-logical axiom that is a premise of the cut has no mean to be simulated at its turn, but this gets rid of lonely non-logical axioms, which is a technical simplification.

DEFINITION 31 (From rewrite rules to L^{nla}). Let $\mathcal{R}$ be a rewrite system. Consider the associated set of non-logical axioms, as in Def. 29 and consider the associated logical system L^{nla} generated by Def. 30.

If $\mathcal{R}$ is confluent, we call L^{nla} a confluent axiomatic system, and call it an atomic system if every antecedent A with $A \rightarrow B \in \mathcal{R}$ is an atomic formula of type o.

Notice that in the logic L^{nla} we do not "rewrite" within the logic directly (as with Deduction Modulo [7, 6]). We apply axiomatic cut rules. We assume we have an atomic and confluent rewrite (or axiomatic) system.

LEMMA 32. *Let $\mathcal{R}$ an atomic and confluent rewrite system. Let $\Gamma \equiv_{\mathcal{R}} \Gamma'$ (pointwise equivalence) be contexts and $A \equiv_{\mathcal{R}} A'$ be formulae. If we have a proof θ of the sequent $\Gamma \vdash^* A$ then we can build a proof of the sequent $\Gamma' \vdash^* A'$.*

Proof. By induction on the structure of θ. We copy every rule, applying the induction hypothesis before that. The only non trivial case are:

- a λ conversion rule. Apply induction hypothesis.
- A' is atomic and we have a logical rule on A. The principle is that we rewrite A' with rewriting cuts (and λ rules) until it becomes non atomic.

  By confluence, we must have a chain of instances of rewrite rules $A_i \rightarrow B_i$, $i \leq n$, such that $A_0 \equiv_\lambda A'$, $A_i \equiv_\lambda B_{i-1}$ and B_n non atomic, having the same main connective (by Lem. 28) as A. Since $B_n \equiv_{\mathcal{R}} A$, we construct a proof of $\Gamma' \vdash^* B_n$ by applying the induction hypothesis. We then add successive rewriting cuts and λ-conversion rules.
- we have the same on the left hand side.
- an axiom rule. We only know the existence $A'' \in \Gamma'$ such that $A'' \equiv_{\mathcal{R}} A \equiv_{\mathcal{R}} A'$, so we can't apply the axiom rule as such since A', A'' are not atomic and the rewriting path between A' and A'' is not straight. But

we know by confluence that there is a formula D such that $A^{''} \to^n D$ and $A' \to^m D$.

We prove by induction over $n+m+\#D$, where $\#D$ is the number of connectives and quantifiers of D that we can build a proof of $A'' \vdash^* A'$. If D is non atomic, while A' or $A^{''}$ is (assume it is A'), let $A' \to B$ the first rule used in the rewrite sequence $A' \to^n D$. We have by induction hypothesis a proof of the sequent $\Gamma' \vdash^* B$ we then make a rewriting cut with the axiom $B \vdash^* A'$, adding λ rules if necessary. Otherwise, if, say $D \equiv_\lambda B_1 \vee B_2$ and A', A'' are compound, they are equal to $B_1' \vee B_2'$ and $B_1^{''} \vee B_2^{''}$. By induction hypothesis we have proofs of $B_1^{''} \vdash^* B_1'$ and $B_2^{''} \vdash^* B_2'$ and we add an $\vee_L$ and an $\vee_R$ rule (plus λ if needed) to conclude. At last, if D itself is atomic, this is just a matter of using cut and λ rules with the non-logical axioms required by the rewriting paths $A' \to^n D$ and $A^{''} \to^m D$.

■

We see that the ability to rewrite atoms with the help of rewriting cuts is essential, as well as confluence.

7 Kleene's lemma and rewriting

7.1 Kleene's lemma in ICTT

Kleene's lemma is a standard rule inversion lemma, saying that – for certain rules – a proof of a sequent that is the conclusion of the rule may be replaced by a proof of the premise of that rule. For instance, if we have a proof of the sequent $\Gamma \vdash^* \forall x A$ then we can construct a proof of the sequent $\Gamma \vdash^* (t/x)A$ for any t. Some rules cannot be inverted, as the $\vee_R$ rule. Indeed, we can find no proof of the sequent $A \vee B \vdash^* A \vee B$ beginning with a $\vee_R$ rule. Other non invertible rules are $\supset_L, \forall_L, \exists_R$. We here prove it in ICTT, in L_{nla} as well as in L^{nla} when the rewrite rules are confluent and atomic. This lemma is used at some places in former sections.

LEMMA 33 (Kleene). *Let $D_1 \equiv_\lambda \ldots \equiv_\lambda D_n \equiv_\lambda B \vee C$. If we have a proof π of the sequent $\Gamma, D_1, \ldots, D_n \vdash^* A$ then we have a proof of $\Gamma, B \vdash^* A$ and $\Gamma, C \vdash^* A$. If the proof was cut-free, then the obtained proofs remain cut-free.*

Proof. Standard, by induction on the height of π (the depth of the associated tree). Notice that if $n = 0$ a use of the weakening rule is sufficient. If the last rule is a rule r on Γ or on A, then apply induction hypothesis and the same rule r. If the rule is an axiom, no D_i is an active formula, so we can replace them by B or C freely. Otherwise, assume D_1 is active. The rule can be:

- a contraction. Apply induction hypothesis on the premise.
- a weakening: do the same.
- a λ-conversion rule. Apply induction hypothesis.
- an $\vee$-l rule, apply induction hypothesis on the premises, we get four proofs, and keep only the two of interest, that we contract.
- an axiom, then the proof has the shape: $\overline{\Gamma, D_1, \ldots, D_n \vdash D_1}$. We expand it into the proof:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, B, D_2, \ldots, D_n \vdash B}\text{Axiom}}{\Gamma, B, D_2, \ldots, D_n \vdash B \vee C}\vee_R}{\Gamma, B, D_2, \ldots, D_n \vdash D_1}\lambda$$

we do the same for C.

Notice that the proof remains valid if we add axiomatic cuts and non-logical axioms. We have two more cases to consider: a non-logical axiom with D_1 as a left (resp. right) active formula. Let's see the left case, with $D_1 \vdash E$ as a non-logical axiom. Then we can construct the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, B, D_2, \ldots, D_n \vdash B}\text{axiom}}{\Gamma, B, D_2, \ldots, D_n \vdash B \vee C}\vee_R}{\Gamma, B, D_2, \ldots, D_n \vdash D_1}\lambda \qquad \dfrac{}{\Gamma, E, \ldots, D_n \vdash E}\text{non-logical axiom}}{\Gamma, B, D_2, \ldots, D_n \vdash E}\text{rewriting cut}$$

introducing a rewriting cut, allowed even in the cut-free case. ■

The lemma and the proof are the same for all the other connectives, save the four mentioned above.

7.2 Kleene's lemma in $\mathbf{L}^{nla}$

Now, we prove Kleene's lemma in the confluent atomic case, in L^{nla}. The statement of the lemma must be somewhat modified to obtain the results we need.

LEMMA 34 (Kleene). *If we have a proof θ of the sequent $\Gamma, D_1, \ldots, D_n \vdash^* A$ then we have a proof of $\Gamma, B \vdash^* A$ and $\Gamma, C \vdash^* A$, where $D_i \equiv_{\mathcal{R}} B \vee C$. If the proofs is cut-free, then the obtained proofs remain cut-free (with only rewriting cuts).*

Proof. The only cases that are different from those of the proof of Lem. 33 are:

- the case of a λ rule is treated by induction hypothesis, even in the case of D_i being an active formula.

- in the inductive cases we have to consider the case of a rewriting cut on Γ, on D_i or on A. Let's consider the third case, and assume the rewriting cut is with the non-logical axiom $E \vdash^* A$. We apply induction hypothesis on the proof of the premise $\Gamma, D_1, \ldots, D_n \vdash^* E$. Then we add a rewriting cut to $\Gamma, B \vdash^* E$. Similarly if the rewriting cut is done with respect to some formula in Γ (first case). If the rewriting cut is done on D_i, then it is sufficient to apply induction hypothesis thanks to the generalized hypothesis.

- the connective case on D_i: thanks to confluence and Lem. 28 it can only be a $\vee_L$ rule. After an application of induction hypothesis, we have proofs of $\Gamma, B, B' \vdash^* A$ and $\Gamma, C, C' \vdash^* A$ with $B' \equiv_{\mathcal{R}} B$ and $C' \equiv_{\mathcal{R}} C$. We then apply Lem. 32 and contract.

- an axiom involving some D_i, say D_1. We have a proof of the sequent $\Gamma, D_1, \ldots, D_n \vdash^* D_1$. By Lem. 32, we have as well a proof of $\Gamma, D_1, D_2, \ldots, D_n \vdash B \vee C$. This is easily transformed into a proof of $\Gamma, B \vdash^* B \vee C$ and $\Gamma, C \vdash^* B \vee C$. We apply Lem. 32 once again to get proofs of $\Gamma, B \vdash^* D_1$ and $\Gamma, C \vdash^* D_1$.

■

8 Atomic confluent rewriting, the sign-preserving case

In this section we show how, by carefully choosing the interpretation of atomic predicates, we can restrict axiomatic cuts to be rewriting cuts in the case of sign-preserving rewrite rules (see Def. 36).

8.1 Sign-preserving condition

DEFINITION 35. Let $\{P_i, i \in \Lambda\}$ the collection of atomic predicate symbols of a language S for ICTT. A decoration on them is a total function $p : \Lambda \to \{+, -\}$. We note P_i^+ (resp. P_i^-) whenever $p(i) = +$ (resp. $p(i) = -$). A formula A in λ normal form is said positive (resp. negative) if and only if it does not contains any flex variable and:

- A is an instantiation of a predicate P_i and $p(i) = +$ (resp. $p(i) = -$).
- $A = \top$ or $A = \bot$ (resp. idem).
- $A = B \wedge C$ and B and C are positive (resp. negative).
- $A = B \vee C$ and B and C are positive (resp. negative).

- $A = B \supset C$ and B is negative and C is positive (resp. B is positive and C is negative).
- $A = \Pi.f$ and for any term t, ft is positive (resp. negative).
- $A = \Sigma.f$ and for any term t, ft is positive (resp. negative).

A formula that is not in λ normal form is said positive (resp. negative) if its corresponding normal form is.

There are many formulae that are neither positive, nor negative. First of all, any formula containing flex variables, as $\forall X.X$ or $P \vee X$, since every instance of X can not have the same polarity. $A \vee \neg A$ is another counterexample. Notice that we even explicitly forbid $\forall X.X$ to be analyzed in Def. 35, although it can be proved that it does not fit the pattern.

Def. 35 may seem a bit circular. It can apply to quantification over propositional types (of type $\ldots \to o$), but only if the bound variable is not at a propositional position. i.e. not “flex”. For instance, $\forall X.P(X)$ is positive provided P is, but not $\forall X.(P(X) \wedge X)$. Since the flex variables are the only impredicative case, Def. 35 is well founded. So we do allow quantification over propositional types, when the bound variable X appears as an argument of a predicate symbol, as for instance $P.X$. With this definition, we will forbid rewrite rules of the form $P(X) \to X$ for any predicate P. Otherwise $\forall X P(X)$ would have a sign, whereas $\forall X.X$ has not. Although this rule seems harmless, it does not fit the pattern for now.

Def. 36 below can be applied as well to rewrite systems.

DEFINITION 36. An axiomatic system is sign-preserving if there exists a decoration on the predicates symbol such that for any axiom $A \vdash B$, A has the same sign than B.

We in this section assume to have a rewrite system being atomic, confluent and sign-preserving, and consider the axiomatic system and the L^{nla} logic associated. The atomicity and confluence conditions implies that Kleene's Lem. 33 holds, which will be of paramount importance in the proof of the main lemma 40.

8.2 Model construction

As in Sec. 4 and 5.1 we can define the complete Heyting algebra Ω_{cfk}, with respect to provability in L^{nla}, here noted $\Omega(\mathrm{L}^{nla})$. We build the same intuitionistic semivaluation π, ν as in Def. 16, and use the Takahashi-Schütte Thm. 8, but with the little following modification in choosing the interpretation of atoms, which is of crucial importance:

DEFINITION 37. Let A be an atomic formula of type o. We let

- $\rho(A) = \nu(A)$ if A is positive.
- $\rho(A) = \pi(A)$ if A is negative.

Since any atomic formula is either positive or negative, this definition is complete. Remember that $\pi(A) = cl(A)$ and $\nu(A) = [\![A]\!]$ are both expressed in L^{nla}. The Takahashi-Schütte Thm. 8 implies then the following definition of the interpretation of formulae:

DEFINITION 38. Let A be a formula of type o. Let ϕ be an environment, associating V-complexes of the right type to variables. Let ϕ^1 the substitution associating the term $\phi(x)^1$ to any variable x. Define A^*_ϕ, the interpretation of A as:

- $A^*_\phi = \nu(\phi^1 A)$ if A is atomic and positive.
- $A^*_\phi = \pi(\phi^1 A)$ if A is atomic and negative.
- otherwise, define it inductively as in Def. 2.

REMARK 39. We could certainly switch the π and ν in the previous two definitions since the choice of polarities is symmetric in Def. 36.

As in Sec. 5.1 it remains to prove only one claim: the model we constructed is a model of the non-logical axioms. Since every time we have $A \vdash B$ as a non-logical axiom we as well have $B \vdash A$, we must show that $A^* = B^*$. The following lemma is the key to show this. It speaks more generally about any positive and negative formulae, but this includes, thanks to the conditions of Def. 36 any non-logical axiom.

LEMMA 40. *Let A a positive (resp. negative) formula. Let ϕ be an environment. Then $(A^*_\phi)^2 = \nu(\phi^1 A)$ (resp. $(A^*_\phi)^2 = \pi(\phi^1 A)$).*

Proof. By induction over the structure of A. Since A does not contain any flex variable (Def. 35), this induction is well-founded. We should consider every single case for A, and we omit to mention the environment ϕ where it is not essential. Also remark that this proof does not work for every π, ν, but only for the one we give (cl and $[\![]\!]$), since they are in a sense maximal.

- if A is atomic, this comes from the very definition of A^*. If A is $\top$, this is because in this particular case $cl(\top) = [\![\top]\!]$, similarly for $\bot$.
- if $A = B \wedge C$ and A is positive, then by induction hypothesis we have $(B^*)^2 = [\![B]\!]$ and $(C^*)^2 = [\![C]\!]$, and then, by definition, $(A^*)^2 = [\![B]\!] \cap [\![C]\!]$. Since $[\![B]\!] \cap [\![C]\!] \leq [\![B \wedge C]\!]$ from Lem. 17 (ν is a upper constraint of an intuitionistic semivaluation), we must show only the converse. Let Γ such that $\Gamma \vdash^* B \wedge C$. Applying Kleene's Lem. 33 gives

us proofs of the sequents $\Gamma \vdash^* B$ and $\Gamma \vdash^* C$. Hence $\Gamma \in [\![B]\!] \cap [\![C]\!]$.

If A is negative, reasoning in the same way leads us to try to show the inclusion $cl(B) \cap cl(C) \subseteq cl(B \wedge C)$. Let $\Gamma \in cl(B) \cap cl(C)$, Γ is B and C-cuttable. Let D such that $B \wedge C \vdash^* D$. Then, $B, C \vdash^* D$ as well, by Kleene's Lem. 33. By B, and then C cuttability, one get a proof of the sequent $\Gamma \vdash^* D$.

- $A = B \vee C$ and A is positive. By the same reasoning as in the previous case, showing that $[\![B \vee C]\!] \subseteq [\![B]\!] \cup [\![C]\!]$ is sufficient. Let Γ be such that $\Gamma \vdash^* B \vee C$. Let also D an upper bound for $[\![B]\!]$ and $[\![C]\!]$: if $\Delta \vdash^* B$ or $\Delta \vdash^* C$ then $\Delta \vdash^* D$. We need to show that $[\![D]\!]$ is an upper bound for $[\![B \vee C]\!]$, i.e. $\Gamma \vdash^* D$.

  We construct a proof of $\Gamma \vdash^* D$ by induction over the proof of $\Gamma \vdash^* E$ with $E \equiv_{\mathcal{R}} B \vee C$. The base case occurs when the active formula is E. If it's a $\bot_R$ rule, then we can as well have this rule generating D instead of E. If it is a rewriting cut or a λ, we simply use the induction hypothesis. If it is a logical rule, it can be only $\vee$-right rule, by confluence. Then we use the hypothesis on D, since we get a proof of either $\Gamma \vdash^* B'$ or $\Gamma \vdash^* C'$, which is the same, by Lem. 32 as a proof of $\Gamma \vdash^* B$ (resp. $\Gamma \vdash^* C$). If it is an axiom, we can, as in the last case of the proof of Lem. 32. add $\vee$-left, $\vee$-right and two axiom rules, at the potential cost of adding rewriting cuts. But this case boils down to the previous ones. For the induction case, the last rule is on a formula of Γ. Apply the induction hypothesis on the premises (if needed, e.g. not on the left premise of $\supset_R$), and apply the same rule. Therefore $\Gamma \in [\![D]\!]$, and it belongs to the least upper bound $[\![B]\!] \cup [\![C]\!]$.

  If A is negative, we show that $cl(B) \cup cl(C) \subseteq cl(B \vee C)$. Let D such that $B \vee C \vdash^* D$. Then by Kleene's Lem. 33, $B \vdash^* D$ and $C \vdash^* D$, and $[\![D]\!]$ is an upper bound for $cl(B)$ and $cl(C)$. Since this holds for any D, the result follows.

- $A = B \supset C$ and A is positive. We prove $[\![B \supset C]\!] \subseteq cl(B) \supset_\Omega [\![C]\!]$, or equivalently $[\![B \supset C]\!] \cap cl(B) \subseteq [\![C]\!]$. Let Γ a B-cuttable context, and such that we have a proof of the sequent $\Gamma \vdash^* B \supset C$. By an application of Kleene's Lem. 33 we have a proof of the sequent $\Gamma, B \vdash^* C$. Since Γ is B-cuttable, it gives a proof of $\Gamma \vdash^* C$. Hence $\Gamma \in [\![C]\!]$.

  If A is negative, then we have to show $[\![B]\!] \supset_\Omega cl(C) \subseteq cl(B \supset C)$. Let $\Gamma \in [\![B]\!] \supset_\Omega cl(C)$, we must show that Γ is $B \supset C$-cuttable.

We can not apply Kleene's Lem. 33, as in the $\vee$ positive case. So, assume we have a proof of $\Delta, E_1, \ldots, E_n \vdash^* D$ for some D, with $E_i \equiv_{\mathcal{R}} B \supset C$. In contrast with the $\vee$ positive case we here have to consider an arbitrary multiplicity n. We show by induction on this proof that we can build a proof of $\Delta, \Gamma \vdash^* D$. As always, if it is a rule on D or a proposition of Δ (including axiom), we apply the induction hypothesis to the premises and then the same rule. Otherwise, if it is a contraction, a weakening, a λ or a rewriting cut on E_i, we apply the induction hypothesis. We forget the axiom case on E_i, since it boils down to the other cases in the same way as in the $\vee$ positive case, an $\supset_R$ rule on, say, E_1. Then we have the proof:

$$\dfrac{\dfrac{\pi_1}{\Delta, E_2, \ldots, E_n \vdash^* B'} \quad \dfrac{\pi_2}{\Delta, E_2, \ldots, E_n, C' \vdash^* D}}{\Delta, E_1, \ldots, E_n \vdash^* D}$$

Applying the induction hypothesis on the premises gives us proofs of the sequents: $\Delta, \Gamma \vdash^* B'$ and $\Delta, \Gamma, C' \vdash^* D$, that we convert into proofs of $\gamma \vdash^* B$ and $\Delta, \Gamma, C \vdash^* D$ by Lem. 32). Hence $\Delta, \Gamma \in [\![B]\!]$, and $\Delta, \Gamma \in cl(\Gamma)$. So by definition of $\supset_\Omega$, $\Delta, \Gamma \in cl(C)$: it is C-cuttable and we have directly a proof of the sequent $\Gamma \vdash^* D$.

- $A = \Pi.f$ and A is positive. Calling $\phi' = \phi + (d/x)$, we know that:

$$A^*_\phi = \bigcap_{d \in D} f^*_\phi d = \bigcap_{d \in D} (fx)^*_{\phi'} = \bigcap_{d \in D} [\![\phi'^1(fx)]\!] = \bigcap_{t \in \mathcal{T}} [\![(\phi'^1 f)t]\!] = \bigcap_{t \in \mathcal{T}} [\![(\phi^1 f)t]\!]$$

the third equality holding by the induction hypothesis, choosing x such that it does not appear in f.

We then show $[\![\phi^1\Pi.f]\!] = [\![\Pi.(\phi^1 f)]\!] \leq \bigcap\{[\![(\phi^1 f).t]\!] \mid t \in \mathcal{T}\}$. Let Γ such that $\Gamma \vdash^* \Pi.\phi^1 f$. Then by Kleene's Lem. 33, we have a proof θ of $\Gamma \vdash^* (\phi^1 f).t$, for any term t. Hence $\Gamma \in \bigcap\{[\![(\phi^1 f).t]\!] \mid t \in \mathcal{T}\}$.

If A is negative, let B be a formula such that $\Pi.(\phi^1 f) \vdash^* B$. We show $\bigcap\{cl((\phi^1 f)t), t \in \mathcal{T}_\alpha\} \subseteq [\![B]\!]$. Let $\Gamma \in \bigcap\{cl((\phi^1 f)t), t \in \mathcal{T}_\alpha\}$. We show that $\Gamma \vdash^* B$, knowing that for any t, and for any C, if $(\phi^1 f)t \vdash^* C$, then $\Gamma \vdash^* C$.

We cannot apply Kleene's Lem. 34. So we construct by induction a proof of $\Gamma, \Delta \vdash^* B$, over the proof structure of a proof of $\Delta, (\Pi.f)^n \vdash^* B$, where n is any number of contractions of $\Pi.f$. Formally we should assume that for $E_i \equiv_{\mathcal{R}} \Pi.f$ as in the $\vee$ positive and $\supset$ negative cases.

$n = 0$ is a trivial case. If the last rule is a rule r on Δ or on B, then apply the induction hypothesis and then r on the proofs we obtain. Otherwise, we then have a proof of $\Delta, ft, (\Pi.f)^{n-1} \vdash^* B$. After applying the induction hypothesis, we get a proof of $\Gamma, \Delta, ft \vdash^* B$. We can then safely replace ft by Γ, since Γ is ft-cuttable by hypothesis. We then contract on the formulae of Γ.

- $A = \Sigma.f$ and A is positive. We must show $[\![\Sigma.(\phi^1 f)]\!] \leq \bigcup\{[\![(\phi^1 f)t]\!], t \in \mathcal{T}\}$, by the same reasoning as in the previous case. We omit ϕ from now on. Let Γ such that $\Gamma \vdash^* \Sigma.f$. We cannot apply Kleene's lemma. Let C such that $[\![ft]\!] \leq [\![C]\!]$ for any term t. We show by induction on the proof of $\Gamma \vdash^* E$, with $E \equiv_{\mathcal{R}} \Sigma.f$ that we can build a proof of $\Gamma \vdash^* C$. As always, copy every left rule, applying the induction hypothesis. And when we get to a right rule, it can be either $\bot_R$, then replace it with $\bot_R$ introducing D, a rewriting cut or a λ, then apply the induction hypothesis, or an axiom, that boils down to the last case, an $\exists_R$ rule. In this case, we get a proof of $\Gamma \vdash^* ft$ for some t. But then, by definition of Γ and C we get directly a proof of $\Gamma \vdash^* C$.

  If A is negative, we show $\bigcup\{cl((\phi^1 f)t)\} \leq cl(\Sigma.(\phi^1 f))$. We omit ϕ. Let Γ such that Γ is ft-cuttable for any t. Assume that we have a proof of $\Sigma.f \vdash^* B$. By Kleene's Lem. 33 we have a proof of $fx \vdash^* B$ for a fresh x. Since Γ is as well fx-cuttable, we have directly $\Gamma \vdash^* B$, and $\Gamma \in cl(\Sigma.f)$.

■

REMARK 41. This lemma just says that the π, ν semivaluation is indeed much more than a semivaluation in the sense of Def. 6. Since we have $\pi(A \wedge B) = \pi(A) \wedge_\Omega \pi(B)$, and so on for the other connectives (special case for $\supset$, since it has a positive and a negative part). We have the same result on ν. This is due to the choice of π, ν we have made. This is not valid any π, ν but for our very specific one.

REMARK 42. Instead of invoking Kleene's lemma, we could use everywhere an induction on the proof structure. Anyway, we would need confluence and atomicity, in order to ensure that the rewrite rules are treated properly.

8.3 Completeness and cut elimination for L^{nla} with sign-preserving axioms

Lem. 40 is stated and proved in the L^{nla} logic. We have carefully chosen the ρ function (Def. 38 and Thm. 8) on the only one degree of freedom we had, the atomic formulae. With this interpretation, we have $A^* = [\![A]\!]$

(resp. $A^* = cl(A)$) in $\Omega(\mathrm{L}^{nla})$, whenever A is positive. This result holds as well for $\Omega(L)$ but we don't care here.

LEMMA 43. *The global model we constructed is a model of the non-logical axioms.*

Proof. If we have, say, a positive non-logical axiom $A \vdash B$ in L^{nla}, we have $[\![A]\!] \subseteq [\![B]\!]$ by an elementary reasoning:

$$\frac{\Gamma \vdash A \qquad \overline{A \vdash B}}{\Gamma \vdash B}$$

The symmetric comes from the symmetric axiom $B \vdash A$ we assumed to have. Hence $A^* = [\![A]\!] = [\![B]\!] = B^*$. We as well have $A^* = cl(A) = cl(B) = B^*$, i.e. Γ is A-cuttable if and only if it is B-cuttable by a similar reasoning.

Therefore, from Lem. 40 for any axiom $A \vdash B$, $A^* = B^*$. ∎

THEOREM 44 (cut-free completeness of L^{nla}). *Consider a set of non-logical axioms atomic, confluent and sign-preserving. Let Γ be a context and A be a formula of S. Assume that for any global model of the non-logical axioms, we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. As Thm. 18 and 25. $\Omega(\mathrm{L}^{nla})$ is a model of the non-logical axioms. ∎

As well, we have the cut elimination theorem as a corollary:

COROLLARY 45 (constructive cut elimination for L^{nla}). *Consider a set of non-logical axioms atomic, confluent and sign-preserving. Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in L^{nla}, then it has a proof in L^{nla} without cut.*

Proof. By the soundness Thm. 23, that remains exactly the same as in Sec. 5, and the above cut-free completeness Thm. 44, both of which were proved constructively. ∎

9 On the constructivity of the proof of cut admissibility

Our proof extends existing semantic proofs for cut admissibility in a number of ways, as remarked above, in particular by adding axioms, and considering the intuitionistic (rather than classical) Theory of Types.

In addition, our proof, unlike [24, 1] for the classical case or [5] for the intuitionistic case, makes no appeal to the excluded middle. The works cited, as ours, start from Schütte's observation [22] that cut admissibility can be proved semantically by showing completeness of the cut-free fragment with respect to semivaluations, and then showing every semivaluation gives rise to a total valuation extending it.

There are a number of pitfalls to avoid here if one wants a constructively valid proof based on this kind of argument, both in the way a semivaluation is produced and how one passes to a valuation.

Andrews shows [1] that any abstract consistency property gives rise to a semivaluation, but then builds one in a way that requires deciding whether or not a refutation exists of a given finite set of sentences. One can also exhibit a semivaluation by developing a tableau refutation of a formula (a Hintikka set) as is done in [5] but must take some care in the way the steps are formalized not to appeal to the fan theorem in order to produce an open path. No discussion of this appears in [5]. Such a step can possibly be done with countable choice, working, say in the realizability interpretation of IZF, but no such formalization has been worked out to the authors' knowledge.

In the proof given above we appeal to the strengthened version in [5] of Schütte's lemma and use the more liberal definition of semivaluation *pairs*. By choosing the context-based semantics developed above we are able to avoid the construction of tableaux, and bypass abstract consistency properties.

As pointed out by Gödel and discussed in e.g. [14, 26], a strictly constructive completeness proof is impossible if an excessively narrow proof of validity is assumed, e.g. conventional Kripke models. However, there are a number of ways to liberalize the definition of validity to "save" constructive completeness [27, 4, 25, 13], in particular by allowing truth-values in arbitrary cHa's or cBa's (complete Heyting/Boolean algebras). In [25] completeness is shown constructively by mapping formulae to their own equivalence class in the Lindenbaum cHa (if the object logic is classical, as in [1] one would use the corresponding Boolean algebra). One is not required decide the provability of formulae in order to show model existence, as one must using just the $\top, \bot$-valued semantics of [1, 24]. The semantics used in this paper is over a similar cHa, modified to work with cut-free proofs. In the final valuation produced, formulae are mapped to sets of contexts that prove them without cut. Here too, one does not have to decide provability to show model existence.

10 Conclusion

We have given a constructive semantic proof of cut-elimination for ICTT, the intuitionistic formulation of Church's Theory of Types introduced by Miller et. al. [16] and various extensions with non-logical axioms, using new techniques extending Takahashi, Schütte and Andrews' original ideas, based on a new formulation of semivaluations from [5] and notions of cut-free context closure and context-based Heyting algebras based on earlier work by Hermant and Okada [19, 18, 3].

The techniques are not especially dependent on the formal systems studied here, and it would be interesting to apply them to other impredicative logics.

Much of the work in the paper on cut-elimination for axiomatic extensions of ICTT is motivated by work in combining rewriting and sequent calculus by Dowek, Werner, Hardin, Kirchner, Hermant and others (deduction modulo) cited earlier in the paper. It is hoped that some of our results could be extended to full higher-order logic modulo, with a characterization of those rewrite rules that preserve cut-elimination.

Olivier Hermant
Univ. Complutense de Madrid, Spain
ohermant@fdi.ucm.es

James Lipton
Wesleyan University, USA
and visiting Researcher,
Univ. Politécnica de Madrid, Spain
jlipton@wesleyan.edu

BIBLIOGRAPHY

[1] Peter Andrews. Resolution in Type Theory. *Journal of Symbolic Logic*, 36(3), 1971.

[2] Franz Baader and Tobias Nipkow. *Term rewriting and all that.* Cambridge University Press, New York, NY, USA, 1998.

[3] Richard Bonichon and Olivier Hermant. On Constructive Cut Admissibility in Deduction Modulo. In *Proceedings of the TYPES conference*. Springer-Verlag, 2007.

[4] H. C.M̃. de Swart. Another Intuitionistic Completeness Proof. *Journal of Symbolic Logic*, 41:644–662, 1976.

[5] M. DeMarco and J. Lipton. Completeness and Cut Elimination in the Intuitionistic Theory of Types. *Journal of Logic and Computation*, pages 821–854, November 2005.

[6] Gilles Dowek, Thérèse Hardin, and Claude Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31:33–72, 2003.

[7] Gilles Dowek and Benjamin Werner. Proof Normalization Modulo. *The Journal of Symbolic Logic*, 68(4):1289–1316, December 2003.

[8] Albert G. Dragalin. *Cut-Elimination Theorem for Higher-order Classical Logic*, pages 243–252. Plenum Press, New York and London, 1987.

[9] Harvey Friedman. Equality between Functionals. In R. Parikh, editor, *Logic Colloquium*, volume 453 of *Lecture Notes in Mathematics*, pages 22–37. Springer, 1975.

[10] J. Y. Girard. Une extension de l'interprétation de Gödel à l'analyse et son application à l'élimination de coupures dans l'analyse et la théorie des types. In J. E. Fenstad, editor, *Proceedings of the second Scandinavian proof theory symposium.* North–Holland, 1971.

[11] Leon Henkin. Completeness in the Theory of Types. *Journal of Symbolic Logic*, 15:81–91, June 1950.

[12] Olivier Hermant. *Méthodes Sémantiques en Déduction Modulo.* PhD thesis, Université Paris 7 – Denis Diderot, 2005.

[13] Georg Kreisel. A Remark on Free Choice Sequences and the Topological Completeness Proofs. *Journal of Symbolic Logic*, 23:369–388, 1958.

[14] Georg Kreisel. On Weak Completeness of Intuitionistic Predicate Logic. *Journal of Symbolic Logic*, 27:139–158, 1962.

[15] H. Läuchli. An Abstract Notion of Realizability for which Intuitionistic Predicate Calculus is Complete. In A. Kino, J. Myhill, and R. E. Vesley, editors, *Intuitionism and Proof Theory*, studies in logic, pages 277–234. North-holland, Amsterdam, London, 1970. Proceedings of the Conference on Intuitionism and Proof Theory, Buffalo, New York, August 1968.

[16] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform Proofs as a Foundation for Logic Programming. *Annals of Pure and Applied Logic*, 51(1-2):125–157, 1991.

[17] John Mitchell. *Foundations for Programming Languages.* MIT Press, Cambridge, Massachusetts, 1996.

[18] Mitsuhiro Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227:333–396, 1999.

[19] Mitsuhiro Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281:471–498, 2002.

[20] Gordon Plotkin. Lambda definability in the full type hierarchy. In J.P. Seldin and J. R. Hindley, editors, *To H.B. Curry: Essays in Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, New York, 1980.

[21] Dag Prawitz. Hauptsatz for Higher Order Logic. *The Journal of Symbolic Logic*, 33(3):452–457, September 1968.

[22] K. Schütte. Syntactical and semantical properties of simple Type Theory. *Journal of Symbolic Logic*, 25:305–326, 1960.

[23] K. Schütte. *Proof Theory.* Springer, 1977.

[24] Moto-o Takahashi. A Proof of Cut-elimination in Simple Type Theory. *J. Math. Soc. Japan*, 19(4), 1967.

[25] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction*, volume 2. Elsevier Science Publishers, 1988.

[26] Dirk van Dalen. *Lectures on Intuitionism*, pages 1 – 94. Number 337 in Lecture Notes in Mathematics. Springer Verlag, 1973.

[27] W. Veldman. An Intuitionistic Completeness Theorem for Intuitionistic Predicate Logic. *Journal of Symbolic Logic*, 41:159–166, 1976.