Double Negation Translations as Morphisms

Frédéric Gilbert[1] and Olivier Hermant[2]

[1] Ecole des Ponts ParisTech, Inria `frederic.a.gilbert@inria.fr`
[2] CRI, MINES ParisTech `olivier.hermant@mines-paristech.fr`

Abstract. Double-negation translations are used to encode classical proofs in intuitionistic logic. We show how those translations can be tuned to meet two targets jointly: base their specifications on the definition of *classical connectives* inside intuitionistic logic, and reduce their impact on the shape and size of formulas, by limiting as much as possible the number of double negations introduced.
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1 Introduction

The relationship between different formal systems is a longstanding field of studies, and involves for instance conservativity, relative consistency or independence problems [2]. As for deductive systems, the natural question is to find a conservative encoding of formulas. By conservative, we mean an encoding of formulas such that a formula is provable in the first system if and only if its encoding is provable in the second system. This work was pioneered by Kolmogorov [10], Gödel [7] and Gentzen [6] for classical and intuitionistic logics. There exist several classes of sequents that are known to be classically provable if and only if they are intuitionistically provable [12].

In this paper, we are interested in defining, inside intuitionistic logic, *classical connectives and quantifiers* so that formulas that are provable in classical logic can be reflected in intuitionistic logic by the *only translation of its connectives and quantifiers* through what is usually called a morphism: a mapping that preserves the structure. It turns out that none of the historical double-negation translations have this property.

Such a mixed logic would allow, in particular, to reason inside the same framework on both classical and intuitionistic formulas, or even to defined mixed formulas, therefore allowing the use of the excluded-middle only on very specific places.

Dowek has introduced a definition of classical connectives that induces a morphism [3], but it has the disadvantage to be very verbose, introducing four negations in front of every subformula, except the whole formula and the atomic ones.

Our contribution is to make such a definition minimalist, in the sense that we introduce as less negations as possible. After having recalled the necessary material in the next section, we show in Sec. 3 how, through a simple idea - the use of De Morgan's laws, we can divide by two the number of negations introduced. Then, in Sec. 4, we exhibit a minimal definition of classical connectives that still yields a correct translation.

2 Prerequisites

We consider any term language $\mathcal{L}$ with variables (denoted x, y) and function symbols, on top of which we build first-order logic formulas with predicate symbols (denoted P, Q), binary connectives $\wedge, \vee, \Rightarrow$, unary connective $\neg$, 0-ary connectives $\top, \bot$ and quantifiers $\forall, \exists$. To avoid many parentheses, we assume that the negation connective $\neg$ binds more tightly than any other connective or quantifier. Multiset of formulas are denoted by Γ, Δ.

As for intuitionistic logic, we also introduce *new, classical, connectives and quantifiers:* $\wedge^c, \vee^c, \Rightarrow^c, \neg^c, \top^c, \bot^c, \forall^c$ and $\exists^c$.

2.1 Deduction Rules

The deduction rules we consider are of two kinds: those of classical sequent calculus LK, given in Fig. 1, and those of intuitionistic sequent calculus LJ, given in Fig. 2, where in the LJ case, the set of formulas Δ is allowed to contain at most one formula. The usual proviso that c must be a fresh constant in $\forall_L$ and $\exists_R$ holds, while t may be any term.

LK enjoys Kleene's inversion lemmas [9,8], stating that inference rules can be permuted and, therefore, gathered. All rules can be permuted at will, except the rules $\exists_R$ and $\forall_L$, that cannot be permuted downwards, and $\forall_R$ and $\exists_L$, that cannot be permuted upwards. Another view of Kleene lemmas is the following: given an LK proof of $\Gamma \vdash A, \Delta$ (resp. $\Gamma, A \vdash \Delta$), we can construct a proof whose first rule applies on A, except if A is an atom, or an existentially (resp. universally) quantified formula. At the same time, modulo slight variations on LK (axiom and weakening rules on atoms only) Kleene's lemmas do not modify the height of the proof. This allows us to use those lemmas while doing induction on the proof height.

In LJ, we will use the following strengthening lemma:

Lemma 1. *Let Γ be a multiset of formulas, and A be a formula. If the sequent $\Gamma, \neg\neg A \vdash$ (resp. the sequent $\Gamma \vdash \neg A$) has a proof, then the sequent $\Gamma, A \vdash$ has a proof.*

Proof. By structural induction on the proofs. If we meet a $\neg_L$ rule on $\neg\neg A$, then we appeal to the induction hypothesis on $\Gamma \vdash \neg A$.

To shorten LJ derivations, we also gather a $\neg_L$ and a $\neg_R$ rule:

Definition 1. *Double-Negation Rules*

$$\neg\neg_L \frac{\Gamma, A \vdash}{\Gamma, \neg\neg A \vdash} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} \neg\neg_R$$

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \mathrm{ax} \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{cut}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot_L \qquad \frac{}{\Gamma \vdash \top, \Delta}\ \top_R$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee_R$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\ \forall_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\ \exists_R$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{contr}_L \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A, A, \Delta}\ \mathrm{contr}_R$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{weak}_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \mathrm{weak}_R$$

Fig. 1. Classical sequent calculus LK

$$\frac{}{\Gamma, A \vdash A}\ \mathrm{ax} \qquad \frac{\Gamma \vdash A \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ \mathrm{cut}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot_L \qquad \frac{}{\Gamma \vdash \top}\ \top_R$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_L \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_R$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_L \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{R1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{R2}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ \Rightarrow_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_R$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash \Delta}\ \neg_L \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}\ \neg_R$$

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}\ \forall_L \qquad \frac{\Gamma \vdash A[c/x]}{\Gamma \vdash \forall x A}\ \forall_R$$

$$\frac{\Gamma, A[c/x] \vdash \Delta}{\Gamma, \exists x A \vdash \Delta}\ \exists_L \qquad \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ \exists_R$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{contr}_L$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \mathrm{weak}_L \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}\ \mathrm{weak}_R$$

Fig. 2. Intuitionistic Sequent Calculus LJ

2.2 Double Negation Translations

In 1925, the first translation, $(.)^K$, is published by Kolmogorov [10]. This translation adds a double negation in front of every connective. In [5], a so-called light version, $(.)^k$ has been introduced, that introduces a double negation in front of every *subformula* of every connective. This avoids to introduce explicitly a double negation on atoms.

$$\begin{array}{rll} A^K & = \neg\neg A \text{ for atoms} & (\neg A)^K = \neg\neg(\neg A^K) \\ \bot^K & = \bot & \top^K = \top \\ (A \wedge B)^K & = \neg\neg(A^K \wedge B^K) & (\forall x A)^K = \neg\neg\forall x A^K \\ (A \vee B)^K & = \neg\neg(A^K \vee B^K) & (\exists x A)^K = \neg\neg\exists x A^K \\ (A \Rightarrow B)^K & = \neg\neg(A^K \Rightarrow B^K) & \end{array}$$

Fig. 3. Kolmogorov's translation

$$\begin{array}{rll} A^k & = A \text{ for atoms} & (\neg A)^k = \neg(\neg\neg A^k) \\ \top^k & = \top & \bot^k = \bot \\ (A \wedge B)^k & = \neg\neg A^k \wedge \neg\neg B^k & (\forall x A)^k = \forall x \neg\neg A^k \\ (A \vee B)^k & = \neg\neg A^k \vee \neg\neg B^k & (\exists x A)^k = \exists x \neg\neg A^k \\ (A \Rightarrow B)^k & = \neg\neg A^k \Rightarrow \neg\neg B^k & \end{array}$$

Fig. 4. Light Kolmogorov's translation

With Kolmogorov's translation, A is provable in LK if and only if A^K is provable in LJ. Since $A^K = \neg\neg A^k$ [5], A is provable in LK if, and only if, $\neg\neg A^k$ is provable in LJ.

2.3 Morphisms

We are looking for *morphisms*:

Definition 2 (Morphism). *Let h be a function that maps classical formulas into intuitionistic formulas. It is a morphism if, and only if, it preserves the connectives, and translates atomic formulas into themselves:*

$$\begin{array}{rlrl} h(P) & = P \quad \textit{for atoms} & & \\ h(\bot) & = \bot^c & h(\top) & = \top^c \\ h(A \wedge B) & = h(A) \wedge^c h(B) & h(A \vee B) & = h(A) \vee^c h(B) \\ h(A \Rightarrow B) & = h(A) \Rightarrow^c h(B) & h(\neg A) & = \neg^c h(A) \\ h(\forall x\ A) & = \forall^c x\ h(A) & h(\exists x\ A) & = \exists^c x\ h(A) \end{array}$$

As well, we want to translate statements and sequents with this morphism, and preserve provability:

Definition 3. *Let Γ be the formulas $A_1, \cdots, A_n$ and A be a formula. We let $h(\Gamma \vdash A) = h(\Gamma) \vdash h(A) = h(A_1), \cdots, h(A_n) \vdash h(A)$.*

Kolmogorov's translation is therefore not a morphism (it introduces double negations on atoms), and its light version, neither, since it only allows to show that $h(\Gamma) \vdash \neg\neg h(A)$.

Dowek's idea [3] was to combine Kolmogorov's original and light translations, to introduce double-negation both at the leaves and at the root of formulas. This amounts to define the classical connectives in intuitionistic logic as in Fig. 5.

This generate a correct morphism, let it be $(.)^D$, in the sense of Thm. 1 below.

$$\begin{array}{rcl} \bot^c = \bot & & \top^c = \top \\ (A \wedge^c B) = \neg\neg(\neg\neg A \wedge \neg\neg B) & & (\forall^c x A) = \neg\neg\forall x \neg\neg A \\ (A \vee^c B) = \neg\neg(\neg\neg A \vee \neg\neg B) & & (\exists^c x A) = \neg\neg\exists x \neg\neg A \\ (A \Rightarrow^c B) = \neg\neg(\neg\neg A \Rightarrow \neg\neg B) & & (\neg^c A) = \neg\neg\neg(\neg\neg A) \end{array}$$

Fig. 5. Classical Connectives in Intuitionistic Logic, following [3]

Theorem 1 **([3]).** *Let Γ be a set of formulas, and A be a formula. If the sequent $\Gamma \vdash A$ is has an LK proof then the sequent $\Gamma^D, \neg A^D \vdash$ has an LJ proof and, if A is not atomic, then the sequent $\Gamma^D \vdash A^D$ has an LK proof. If the sequent $\vdash A$ has an LK proof, then the sequent $\vdash A^D$ has an LJ proof.*

As we see, the morphism $(.)^D$ introduces twice as much negations as Kolmogorov's, that was already quite verbose.

3 A Lighter Morphism with De Morgan's Laws

We follow the line of De Morgan's law, that are valid in classical logic. Since they introduce a negation on top of the connective and a negation inside the connective, we are guaranteed that double negation will appear at the root and the leaves of formulas, which is sufficient to show Thm. 1.

3.1 The Classical Connectives

Blindly applying this idea, introduces one negation before every connective, and a negation on the subformula(s) of the connective. For instance, $A \vee^c B$ will be $\neg(\neg A \wedge \neg B)$, while $\neg^c A$ will be $\neg\neg\neg A$. But we can pluck low-hanging fruits: in intuitionistic logic, $\neg\neg\neg A$ is equivalent to $\neg A$, and as well, we can remove double negations on the two 0-ary connectives $\top$ and $\bot$.

Definition 4 (De Morgan's Classical Connectives).

$$\begin{array}{rclrcl} \bot^c & = & \bot & \top^c & = & \top \\ A \wedge^c B & = & \neg(\neg A \vee \neg B) & A \vee^c B & = & \neg(\neg A \wedge \neg B) \\ A \Rightarrow^c B & = & \neg(\neg\neg A \wedge \neg B) & \neg^c A & = & \neg A \\ \forall^c x A & = & \neg\exists x \neg A & \exists^c x A & = & \neg\forall x \neg A \end{array}$$

As an example, the formula $A \wedge (B \Rightarrow C)$. It is translated into:

$$(A \wedge (B \Rightarrow C))^\circ = \neg(\neg(A)^\circ \vee \neg(B \Rightarrow C)^\circ) = \neg(\neg A \vee \neg\neg(\neg\neg(B)^\circ \wedge \neg(C)^\circ)) = \neg(\neg A \vee \neg\neg(\neg\neg B \wedge \neg C))$$

3.2 Translating Proofs

Theorem 2. *Given a classical proof of the sequent $\Gamma \vdash \Delta$, we can build an intuitionistic proof of the sequent $(\Gamma)^\circ, \neg(\Delta)^\circ \vdash$*

Proof. By induction on the proof of $\Gamma \vdash \Delta$, considering its last rule:

- axiom. We translate it as:

$$\neg_L\,\dfrac{\text{ax}\,\dfrac{}{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash (A)^\circ}}{(\Gamma)^\circ, A, \neg(A)^\circ, \neg(\Delta)^\circ \vdash}$$

- cut. We translate it as:

$$\text{cut}\,\dfrac{\neg_R\,\dfrac{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg(A)^\circ} \qquad (\Gamma)^\circ, \neg(A)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash}$$

- $\bot_L$ and $\top_R$ are translated by themselves:

$$\dfrac{}{(\Gamma)^\circ, \bot, \neg(\Delta)^\circ \vdash}\,\bot_L \qquad\qquad \neg_L\,\dfrac{\top_R\,\dfrac{}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \top}}{(\Gamma)^\circ, \neg\top, \neg(\Delta)^\circ \vdash}$$

- $\wedge_L$ rule. We translate it as:

$$\text{contr}_L\,\dfrac{\neg_L\,\dfrac{\vee_{R1}\,\dfrac{\neg_R\,\dfrac{\neg_L\,\dfrac{\vee_{R2}\,\dfrac{\neg_R\,\dfrac{(\Gamma)^\circ, (A)^\circ, (B)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash \neg(B)^\circ}}{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash \neg(A)^\circ \vee \neg(B)^\circ}}{(\Gamma)^\circ, (A)^\circ, \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash}}{(\Gamma)^\circ, \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash \neg(A)^\circ}}{(\Gamma)^\circ, \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash \neg(A)^\circ \vee \neg(B)^\circ}}{(\Gamma)^\circ, \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash}}{(\Gamma)^\circ, \neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash}$$

- $\wedge_R$ rule. We translate it as:

$$\neg\neg_L\,\dfrac{\vee_L\,\dfrac{(\Gamma)^\circ, \neg(A)^\circ, \neg(\Delta)^\circ \vdash \qquad (\Gamma)^\circ, \neg(B)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(A)^\circ \vee \neg(B)^\circ, \neg(\Delta)^\circ \vdash}}{(\Gamma)^\circ, \neg\neg(\neg(A)^\circ \vee \neg(B)^\circ), \neg(\Delta)^\circ \vdash}$$

- $\vee_L$ rule. After applying induction hypothesis, and getting two proofs π_1' and π_2' we translate it as:

$$\neg_L\,\dfrac{\wedge_R\,\dfrac{\neg_R\,\dfrac{\dfrac{\pi_1'}{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash}}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg(A)^\circ} \qquad \neg_R\,\dfrac{\dfrac{\pi_2'}{(\Gamma)^\circ, (B)^\circ, \neg(\Delta)^\circ \vdash}}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg(B)^\circ}}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg(A)^\circ \wedge \neg(B)^\circ}}{(\Gamma)^\circ, \neg(\neg(A)^\circ \wedge \neg(B)^\circ), \neg(\Delta)^\circ \vdash}$$

From now on, we omit to mention the application of induction hypothesis, which remains implicit.

– $\vee_R$ rule. We translate it as:

$$\dfrac{\dfrac{(\Gamma)^\circ, \neg(A)^\circ, \neg(B)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(A)^\circ \wedge \neg(B)^\circ, \neg(\Delta)^\circ \vdash}\wedge_L}{(\Gamma)^\circ, \neg(\neg(A)^\circ \wedge \neg(B)^\circ), \neg(\Delta)^\circ \vdash}\neg\neg_L$$

– $\Rightarrow_L$ rule. We translate it as:

$$\dfrac{\dfrac{\dfrac{(\Gamma)^\circ, \neg(A)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg\neg(A)^\circ}\neg_R \quad \dfrac{(\Gamma)^\circ, (B)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg(B)^\circ}\neg_R}{(\Gamma)^\circ, \neg(\Delta)^\circ \vdash \neg\neg(A)^\circ \wedge \neg(B)^\circ}\wedge_R}{(\Gamma)^\circ, \neg(\neg\neg(A)^\circ \wedge \neg(B)^\circ), \neg(\Delta)^\circ \vdash}\neg_L$$

– $\Rightarrow_R$ rule. We translate it as:

$$\dfrac{\dfrac{\dfrac{(\Gamma)^\circ, (A)^\circ, \neg(B)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg\neg(A)^\circ, \neg(B)^\circ, \neg(\Delta)^\circ \vdash}\neg\neg_L}{(\Gamma)^\circ, \neg\neg(A)^\circ \wedge \neg(B)^\circ, \neg(\Delta)^\circ \vdash}\wedge_L}{(\Gamma)^\circ, \neg\neg(\neg\neg(A)^\circ \wedge \neg(B)^\circ), \neg(\Delta)^\circ \vdash}\neg\neg_L$$

– $\neg_L$ rule. We do not translate it, as $(\neg A)^\circ = \neg(A)^\circ$.

– $\neg_R$ rule. We translate it as:

$$\dfrac{(\Gamma)^\circ, (A)^\circ, \neg(\Delta)^\circ \vdash}{(\Gamma)^\circ, \neg(\neg A)^\circ, \neg(\Delta)^\circ \vdash}\neg\neg_L$$

– universal and existential quantifier rules follow the pattern given by conjunction and disjunction, respectively.

– structural rules (contr, weak) are transformed into the corresponding structural rules on the left-hand side.

3.3 Corollaries

We enjoy the same corollaries as in [3], namely:

Corollary 1. *If the sequent $\Gamma \vdash A$ has an LK proof, and A is not an atomic formula, then the sequent $(\Gamma)^\circ \vdash (A)^\circ$ has an LJ proof.*

Proof. From Thm. 2 we have a proof of $(\Gamma)^\circ, \neg(A)^\circ \vdash$.

If A is $\top$, we can build a trivial LJ proof, applying the $\neg_L$ and $\top_R$ rules, if A is $\bot$, then we can remove it from the classical proof, by replacing axioms with $\bot$ by the $\bot_L$ rule and we get by Thm. 2 an LJ proof of $(\Gamma)^\circ \vdash$, that we turn with a weakening into a proof of $(\Gamma)^\circ \vdash \bot$.

Otherwise, $(A)^\circ$ is of the form $\neg B$, for some B. Therefore, we have a proof of the sequent $(\Gamma)^\circ, \neg\neg B \vdash$, that we turn into a proof of $(\Gamma)^\circ, B \vdash$ with Lem. 1. We then apply the $\neg_R$ rule to yield the desired proof.

Corollary 2. *If the sequent $\vdash A$ has an LK proof, then the sequent $\vdash (A)^\circ$ has an LJ proof.*

Proof. A cannot be atomic, for the sequent $\vdash A$ would have no proof in LK. Cor. 1 prevails.

4 A Minimal Translation

We consider the following definition of classical connectives, and the associated morphism $(.)^m$.

Definition 5 (Minimal Classical Connectives).

$$\begin{array}{rclcrcl} \bot^c & = & \bot & \quad & \top^c & = & \top \\ A \wedge^c B & = & A \wedge B & & A \vee^c B & = & \neg\neg(A \vee B) \\ A \Rightarrow^c B & = & A \Rightarrow \neg\neg B & & \neg^c A & = & \neg A \\ \forall^c x A & = & \forall x \neg\neg A & & \exists^c x A & = & \neg\neg\exists x A \end{array}$$

4.1 Minimality

Theorem 3. *None of the morphisms obtained by removing one of the introduced double negations is a correct translation from LK to LJ.*

Proof. For each morphism φ that does not introduce a double-negation at at least one place where $(.)^m$ does, we provide a counter-example, i.e. a formula A such that:

- A can be proved in LK
- $\varphi(A)$ cannot be proved in LJ

We let P be a atomic predicate symbol, and only give the counterexamples. Their unprovability in LJ is left to the reader:

- if $\varphi(A \vee B) = \varphi(A) \vee \varphi(B)$. $C = P \vee \neg P$ can be proved in LK, and $\varphi(C)$ cannot be proved in LJ.
- if $\varphi(A \Rightarrow B) = \varphi(A) \Rightarrow \varphi(B)$. $C = \neg\neg P \Rightarrow P$ can be proved in LK, and $\varphi(C)$ cannot be proved in LJ.
- if $\varphi(\forall x A) = \forall x \varphi(A)$. $B = \forall x \neg\neg P(x) \Rightarrow \forall x P(x)$ can be proved in LK, but $\forall x \neg\neg P(x) \Rightarrow \neg\neg\forall x P(x)$ cannot be proved in LJ. In consequence, $\varphi(B)$ cannot be proved, whatever the translation of $\Rightarrow$ and $\neg$ we have.
- if $\varphi(\forall x A) = \neg\neg\forall x \varphi(A)$. $B = \exists y(\neg P(y) \Rightarrow \neg\forall x P(x))$ is provable in LK, but $\neg\neg\exists y(\neg P(y) \Rightarrow \neg\neg\forall x P(x))$ is not provable in LJ. In consequence, $\varphi(B)$ cannot be proved, whatever the translation of $\neg$, $\Rightarrow$ and $\exists$ we have.
- $\varphi(\exists x A) = \exists x \varphi(A)$. $\exists x(P(x) \Rightarrow \forall y P(y))$ can be proved in LK, but $\exists x(P(x) \Rightarrow \neg\neg\forall y \neg\neg P(y))$ cannot be proved in LJ.

4.2 Translating Proofs

We first need some results in intuitionistic logic. Notice the similarity of some of the laws we state with De Morgan's translation of Sec. 3:

Lemma 2. *Let Γ be a set of formulas, and A and B be formulas. The following sequents can be proved in LJ:*

$$\begin{array}{lcllcl} \Gamma, \neg(A \wedge B) & \vdash & \neg(\neg\neg A \wedge \neg\neg B) & \qquad \Gamma, \neg(A \vee B) & \vdash & \neg A \wedge \neg B \\ \Gamma, A \Rightarrow \neg\neg B & \vdash & \neg(\neg\neg A \wedge \neg B) & \qquad \Gamma, \neg\exists x A & \vdash & \neg A[t/x] \end{array}$$

Proof. The proofs are the following:

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma, A, B \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A, B \vdash B$}
\LeftLabel{$\wedge_R$}
\BinaryInfC{$\Gamma, A, B \vdash A \wedge B$}
\LeftLabel{$\neg_L$}
\UnaryInfC{$\Gamma, \neg(A \wedge B), A, B \vdash$}
\LeftLabel{$\neg\neg_L$}
\UnaryInfC{$\Gamma, \neg(A \wedge B), A, \neg\neg B \vdash$}
\LeftLabel{$\neg\neg_L$}
\UnaryInfC{$\Gamma, \neg(A \wedge B), \neg\neg A, \neg\neg B \vdash$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$\Gamma, \neg(A \wedge B), \neg\neg A \wedge \neg\neg B \vdash$}
\LeftLabel{$\neg_R$}
\UnaryInfC{$\Gamma, \neg(A \wedge B) \vdash \neg(\neg\neg A \wedge \neg\neg B)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma, A \vdash A$}
\LeftLabel{$\vee_{R1}$}
\UnaryInfC{$\Gamma, A \vdash A \vee B$}
\RightLabel{$\neg_L$}
\UnaryInfC{$\Gamma, \neg(A \vee B), A \vdash$}
\LeftLabel{$\neg_R$}
\UnaryInfC{$\Gamma, \neg(A \vee B) \vdash \neg A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, B \vdash B$}
\RightLabel{$\vee_{R2}$}
\UnaryInfC{$\Gamma, B \vdash A \vee B$}
\RightLabel{$\neg_L$}
\UnaryInfC{$\Gamma, \neg(A \vee B), B \vdash$}
\RightLabel{$\neg_R$}
\UnaryInfC{$\Gamma, \neg(A \vee B) \vdash \neg B$}
\LeftLabel{$\wedge_R$}
\BinaryInfC{$\Gamma, \neg(A \vee B) \vdash \neg A \wedge \neg B$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma, A, \neg B \vdash A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, A, \neg B \vdash \neg B$}
\LeftLabel{$\neg_L$}
\UnaryInfC{$\Gamma, \neg\neg B, A, \neg B \vdash$}
\LeftLabel{$\Rightarrow_L$}
\BinaryInfC{$\Gamma, A \Rightarrow \neg\neg B, A, \neg B \vdash$}
\LeftLabel{$\neg\neg_L$}
\UnaryInfC{$\Gamma, A \Rightarrow \neg\neg B, \neg\neg A, \neg B \vdash$}
\LeftLabel{$\wedge_L$}
\UnaryInfC{$\Gamma, A \Rightarrow \neg\neg B, \neg\neg A \wedge \neg B \vdash$}
\LeftLabel{$\neg_R$}
\UnaryInfC{$\Gamma, A \Rightarrow \neg\neg B \vdash \neg(\neg\neg A \wedge \neg B)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Gamma, A[t/x] \vdash A[t/x]$}
\LeftLabel{$\exists_R$}
\UnaryInfC{$\Gamma, A[t/x] \vdash \exists x A$}
\LeftLabel{$\neg_L$}
\UnaryInfC{$\Gamma, \neg\exists x A, A[t/x] \vdash$}
\LeftLabel{$\neg_R$}
\UnaryInfC{$\Gamma, \neg\exists x A \vdash \neg A[t/x]$}
\end{prooftree}$$

Theorem 4. *Let Γ and Δ be two multisets of formulas. Given a proof of the sequent $\Gamma \vdash \Delta$ in LK, we can build an proof of the sequent $(\Gamma)^m, \neg(\Delta)^m \vdash$ in LJ.*

Proof. By induction on the proof of $\Gamma \vdash \Delta$. In the following, the notation Σ will be used as an abbreviation of $(\Gamma)^m, \neg(\Delta)^m$. Consider the last rule of the proof:

- axiom. We translate it as:

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\Sigma, (A)^m \vdash (A)^m$}
\LeftLabel{$\neg_L$}
\UnaryInfC{$\Sigma, (A)^m, \neg(A)^m \vdash$}
\end{prooftree}$$

- cut. We translate it as:

$$\begin{prooftree}
\AxiomC{$\Sigma, (A)^m \vdash$}
\LeftLabel{$\neg_R$}
\UnaryInfC{$\Sigma \vdash \neg(A)^m$}
\AxiomC{$\Sigma, \neg(A)^m \vdash$}
\LeftLabel{cut}
\BinaryInfC{$\Sigma \vdash$}
\end{prooftree}$$

- $\bot_L$ is translated as itself.
- $\top_R$. We translate it as:

$$\begin{prooftree}
\AxiomC{}
\LeftLabel{$\top_R$}
\UnaryInfC{$\Sigma \vdash \top$}
\LeftLabel{$\neg_L$}
\UnaryInfC{$\Sigma, \neg\top \vdash$}
\end{prooftree}$$

– $\wedge_L$ rule. We translate it as:

$$\wedge_L\ \dfrac{\Sigma, (A)^m, (B)^m \vdash}{\Sigma, (A)^m \wedge (B)^m \vdash}$$

– $\wedge_R$ rule. We translate it as:

$$\text{cut}\ \dfrac{T \qquad \text{weak}_L\ \dfrac{\wedge_R\ \dfrac{\wedge_R\ \dfrac{\neg_R\ \dfrac{\Sigma, \neg(A)^m \vdash}{\Sigma, \vdash \neg\neg(A)^m} \qquad \neg_R\ \dfrac{\Sigma, \neg(B)^m \vdash}{\Sigma, \vdash \neg\neg(B)^m}}{\Sigma \vdash \neg\neg(A)^m \wedge \neg\neg(B)^m}}{\Sigma, \neg(\neg\neg(A)^m \wedge \neg\neg(B)^m) \vdash}}{\Sigma, \neg((A)^m \wedge (B)^m), \neg(\neg\neg(A)^m \wedge \neg\neg(B)^m) \vdash}}{\Sigma, \neg((A)^m \wedge (B)^m) \vdash}$$

where T is obtained from Lem. 2.

– $\vee_L$ rule. After applying induction hypothesis, and getting two proofs π'_1 and π'_2, we translate it as:

$$\neg\neg_L\ \dfrac{\vee_L\ \dfrac{\dfrac{\pi'_1}{\Sigma, (A)^m \vdash} \qquad \dfrac{\pi'_2}{\Sigma, (B)^m \vdash}}{\Sigma, (A)^m \vee (B)^m \vdash}}{\Sigma, \neg\neg((A)^m \vee (B)^m) \vdash}$$

From now on, we omit to mention the application of induction hypothesis, which remains implicit.

– $\vee_R$ rule. We translate it as:

$$\neg\neg_L\ \dfrac{\text{cut}\ \dfrac{T \qquad \text{weak}_L\ \dfrac{\wedge_L\ \dfrac{\Sigma, \neg(A)^m, \neg(B)^m \vdash}{\Sigma, \neg(A)^m \wedge \neg(B)^m \vdash}}{\Sigma, \neg((A)^m \vee (B)^m), \neg(A)^m \wedge \neg(B)^m \vdash}}{\Sigma, \neg((A)^m \vee (B)^m) \vdash}}{\Sigma, \neg\neg\neg((A)^m \vee (B)^m) \vdash}$$

where T is obtained from Lem. 2.

– $\Rightarrow_L$ rule. We translate it as:

$$\text{cut}\ \dfrac{T \qquad \text{weak}_L\ \dfrac{\neg_L\ \dfrac{\wedge_R\ \dfrac{\neg_R\ \dfrac{\Sigma, \neg(A)^m \vdash}{\Sigma \vdash \neg\neg(A)^m} \qquad \neg_R\ \dfrac{\Sigma, (B)^m \vdash}{\Sigma \vdash \neg(B)^m}}{\Sigma \vdash \neg\neg(A)^m \wedge \neg(B)^m}}{\Sigma, \neg(\neg\neg(A)^m \wedge \neg(B)^m) \vdash}}{\Sigma, (A)^m \Rightarrow \neg\neg(B)^m, \neg(\neg\neg(A)^m \wedge \neg(B)^m) \vdash}}{\Sigma, (A)^m \Rightarrow \neg\neg(B)^m \vdash}$$

where T is obtained from Lem. 2.

– $\Rightarrow_R$ rule. We translate it as:

$$\neg_L\,\dfrac{\Rightarrow_R\,\dfrac{\neg_R\,\dfrac{\Sigma, (A)^m, \neg(B)^m \vdash}{\Sigma, (A)^m \vdash \neg\neg(B)^m}}{\Sigma \vdash (A)^m \Rightarrow \neg\neg(B)^m}}{\Sigma, \neg((A)^m \Rightarrow \neg\neg(B)^m) \vdash}$$

- $\neg_L$ is not translated, as in Thm. 2.
- $\neg_R$ rule. We translate it as:

$$\neg\neg_L\,\dfrac{\Sigma, (A)^m \vdash}{\Sigma, \neg\neg(A)^m \vdash}$$

- $\forall_L$ rule. We translate it as:

$$\forall_L\,\dfrac{\neg\neg_L\,\dfrac{\Sigma, (A[t/x])^m \vdash}{\Sigma, \neg\neg(A[t/x])^m \vdash}}{\Sigma, \forall x \neg\neg(A)^m \vdash}$$

- $\forall_R$ rule. We translate it as:

$$\neg_L\,\dfrac{\forall_R\,\dfrac{\neg_R\,\dfrac{\Sigma, \neg(A[c/x])^m \vdash}{\Sigma \vdash \neg\neg(A[c/x])^m}}{\Sigma \vdash \forall x \neg\neg(A)^m}}{\Sigma, \neg\forall x \neg\neg(A)^m \vdash}$$

- $\exists_L$ rule. We translate it as:

$$\neg\neg_L\,\dfrac{\exists_L\,\dfrac{\Sigma, (A[c/x])^m \vdash}{\Sigma, \exists x (A)^m \vdash}}{\Sigma, \neg\neg\exists x (A)^m \vdash}$$

- $\exists_R$ rule. We translate it as:

$$\neg\neg_L\,\dfrac{\text{cut}\,\dfrac{T \qquad \text{weak}_L\,\dfrac{\Sigma, \neg(A[t/x])^m \vdash}{\Sigma, \neg\exists x (A)^m, \neg(A[t/x])^m \vdash}}{\Sigma, \neg\exists x (A)^m \vdash}}{\Sigma, \neg\neg\neg\exists x (A)^m \vdash}$$

  where T is obtained from Lem. 2.
- structural rules (contr, weak) are transformed into the corresponding structural rules on the left-hand side.

4.3 Corollaries

With this minimal translation, we lose some expressiveness with respect to the De Morgan translation discussed in Sec. 3. For instance, the sequent $\neg\neg(A \wedge B) \vdash A \wedge B$ is provable in LK, but its translation, $\neg\neg((A)^m \wedge (B)^m) \vdash (A)^m \wedge (B)^m$ is not necessarily provable in LJ.

Analyzing the situation, we remark that this is because we do not know whether double-negations have been introduced on the right-hand side, and we need at least one. If A or B is atomic, then the intuitionistic sequent is unprovable. If, in front of $(A)^m$ and $(B)^m$, there is a double negation, then we are safe. But this double-negation can also appear deeper in the formula. This is the notion of *barred formula*:

Definition 6 (Barred Formula). *A formula A is said barred if it is not an atomic formula and:*

- *its main connective is $\neg, \vee, \Rightarrow, \top, \bot$ or its has a main quantifier ($\exists, \forall$)*
- *its main connective is $\wedge$ and its immediate subformulas are barred.*

We can prove:

Theorem 5. *Let A be a barred formula, Γ be a set of formula. The sequent $\Gamma \vdash A$ has a proof in LK if, and only if, $(\Gamma)^m \vdash (A)^m$ has a proof in LJ.*

Informally, we have a "bar of negations" on $(A)^m$ (seen as a tree), and we use Kleene's inversion lemma to decompose it until we reach the bar.

Proof. We show the only if part, since the if part follows from the fact that, for any B, B is equiprovable with $(B)^m$ in LK. The proof is done by induction on the structure of A:

- if A is a barred formula with main connective $\neg, \vee$, or main connective $\exists$, then $(A)^m = \neg\neg B$. We apply Thm. 4, that yields a proof of $(\Gamma)^m, \neg\neg\neg B \vdash$, then Lem. 1 to get a proof of $(\Gamma)^m, \neg B \vdash$, that we finally turn into a proof of $(\Gamma)^m \vdash (A)^m$ with a $\neg_R$ rule.
- if A is $\top$, the sequent $(\Gamma)^m \vdash (\top)^m$ has a trivial LJ proof. If A is $\bot$, the LK proof of the sequent $\Gamma \vdash \bot$ can be turned into a proof of the sequent $\Gamma \vdash$, by replacing axioms with $\bot$ by the $\bot_L$ rule. Then we apply Thm. 4 and weak$_R$.
- if $A = B \Rightarrow C$, we apply Kleene's inversion lemma, and get an LK proof of $\Gamma, B \vdash C$. We apply Thm. 4, and get a proof of $(\Gamma)^m, (B)^m, \neg(C)^m \vdash$. Given that $(B \Rightarrow C)^m = (B)^m \Rightarrow \neg\neg(C)^m$, we build the following proof:

$$\cfrac{\cfrac{(\Gamma)^m, (B)^m, \neg(C)^m \vdash}{(\Gamma)^m, (B)^m \vdash \neg\neg(C)^m}\,\neg_R}{(\Gamma)^m \vdash (B \Rightarrow C)^m}\,\Rightarrow_R$$

- if $A = \forall x B$, we apply Kleene's inversion lemma and get an LK proof of $\Gamma \vdash B[c/x]$.We apply Thm. 4 to get a LJ proof of $(\Gamma)^m, \neg(B[c/x])^m \vdash$. Then, we build the following proof:

$$\dfrac{\dfrac{(\Gamma)^m, \neg(B[c/x])^m \vdash}{(\Gamma)^m \vdash \neg\neg(B[c/x])^m}\neg_R}{(\Gamma)^m \vdash (\forall x B)^m}\forall_R$$

– if $A = B \wedge C$, we apply Kleene's inversion lemma, and get two LK proofs of $\Gamma \vdash B$ and $\Gamma \vdash C$, on which we apply induction hypothesis. We combine them with the $\wedge_R$ rule as follows:

$$\dfrac{(\Gamma)^m \vdash (A)^m \qquad (\Gamma)^m \vdash (B)^m}{(\Gamma)^m \vdash (A)^m \wedge (B)^m}\wedge_R$$

Interestingly, if the context Γ is empty, Thm. 5 can be extended to all formulas:

Corollary 3. *Let A be a formula. The sequent $\vdash A$ has a proof in LK if, and only if, $\vdash (A)^m$ has a proof in LJ.*

One could prove this corollary by induction (see the annex), but this is an immediate consequence of Thm. 5 applied to barred formulas, since no barred formulas has a proof in the empty context in LK.

Lemma 3. *Let A be a formula that is not a barred formula. Then the sequent $\vdash A$ has no proof in LK.*

Proof. By induction on the structure of A. For a contradiction, assume that we have a proof of $\vdash A$. If A is atomic, it cannot have an LK proof within an empty context. If A is not atomic, $A = B \wedge C$ since it is not barred. By Kleene's inversion lemma, we have proofs of the sequents $\vdash B$ and $\vdash C$. But one of B and C is not barred, assume it is B. We apply induction hypothesis on B, yielding a contradiction.

5 Conclusion

We have introduced light translations, that turn a formula into a counterpart formula, such that:

– they are correct: if the original formula is provable in classical logic, then its translation is provable in intuitionistic logic.
– it is a morphism: the translations turns connectives into *other connectives*, and sequents into the same sequents.

Moreover, the morphism that we have defined in Sec. 4 is minimal, in the sense that we cannot remove any double negation.

There is still much work to do. First of all, the proof of Thm. 4 appeals to the cut rule. This is not critical in our settings because we translate proofs with cuts, but if we want to preserve cut-freeness, then we cannot use Thm. 4. However, we still can use Thm. 2, that does not introduce any new cut, but the price to pay is the minimality of the morphism. Of course, since the cut rule is admissible, it is always possible to eliminate cuts and Thm. 4 *must* be provable with no appeal to the cut rule.

Technically speaking, such a proof should make a heavy use of Kleene's inversion lemma in LK, and extensions: we need to make rules commute upwards, similarly to a syntactic cut elimination procedure. The tactic here will be to gather rules on the same formula, in other words, to ensure that the next LK rule applies once again on the same formula (or on its subformula). This is exactly what focusing is talking about [11], so that, rather to translate proofs of LK, we should focus on translating focused proofs.

We also intend to extend those morphisms to more powerful frameworks, in particular to Deduction Modulo [4], where we will need to translate rewrite rules as well, and to higher-order logics [1].

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6 Annex 1

We prove, in a direct way, Lem. 3:

Lemma 4. *If the sequent* $\vdash A$ *has a proof in LK, then the sequent* $\vdash (A)^m$ *has a proof in LJ.*

Proof. By induction on the structure formula A:

- A is a predicate:
  No rule can be applied to prove $\vdash A$, so this sequent has no proof in LK
- $\bot$:
  No rule can be applied to prove $\vdash A$, so this sequent has no proof in LK
- $\top$:
  We consider the LJ proof

$$\top_R \frac{}{\vdash \top}$$

- $A \wedge B$:
  We get from Kleene's inversion lemma proofs of $\vdash A$ and $\vdash B$ in LK. Applying the induction hypothesis, we can consider the LJ proof

$$\wedge_R \frac{\vdash (A)^m \qquad \vdash (B)^m}{\vdash (A)^m \wedge (B)^m}$$

- $A \vee B$:
  Applying Thm. 4, we get a LJ proof of:

$$\neg\neg\neg((A)^m \vee (B)^m) \vdash$$

  Then we get from Kleen's inversion lemma a LJ proof of:

$$\vdash \neg\neg((A)^m \vee (B)^m)$$

- $A \Rightarrow B$:
  We get from Kleene's inversion lemma a proof of $A \vdash B$ in LK. Applying the induction hypothesis, we can consider the LJ proof

$$\Rightarrow_R \frac{\neg_R \dfrac{(A)^m, \neg(B)^m \vdash}{(A)^m \vdash \neg\neg(B)^m}}{\vdash (A)^m \Rightarrow \neg\neg(B)^m}$$

- $\neg A$:
  Applying Thm. 4, we get a LJ proof of:

$$\neg\neg(A)^m \vdash$$

  Then we get from Kleen's inversion lemma a LJ proof of:

$$\vdash \neg(A)^m$$

– $\forall x A$:
We get from Kleene's inversion lemma a proof of $\vdash A[C/x]$ in LK. Applying the induction hypothesis, we can consider the LJ proof

$$\forall_R \dfrac{\neg\neg_R \dfrac{\vdash (A[c/x])^m}{\vdash \neg\neg(A[c/x])^m}}{\vdash \forall x \neg\neg(A)^m}$$

– $\exists x A$:
Applying Thm. 4, we get a LJ proof of:

$$\neg\neg\neg(\exists A)^m \vdash$$

Then we get from Kleene's inversion lemma a LJ proof of:

$$\vdash \neg\neg(\exists A)^m$$